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Journal of Algebra 277 (2004) 769–805

JOURNAL OF
Algebrawww.elsevier.com/locate/jalgebra

Unital associative pseudoalgebras and their representations

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Received 7 July 2002

Available online 24 October 2003

Communicated by Efim Zelmanov

Abstract

Pseudoalgebras, introduced in [B. Bakalov, A. D’Andrea, V.G. Kac, Adv. Math. 162 (2001) 1–140], are multi-dimensional analogues of conformal algebras, which provide an axiomatic description of the singular part of the operator product expansion.

Our main interest in this paper is the pseudoalgebra Cend_n , which is the analogue of an algebra of endomorphisms of a finite module. We study its algebraic properties. In particular, we introduce the class of unital pseudoalgebras and describe their structure and representations. Also, we classify pseudoalgebras algebraically similar to Cend_n .

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1. Introduction

Recent years saw several new approaches in the theory of vertex algebras. One was to provide an axiomatic description of the “singular” part of the vertex algebra (which came to be called a conformal algebra), thus describing the vertex algebra as a highest-weight module [10]. The other strives for a coordinate-less description of vertex algebras, in particular leading to the construction of analogous structures over any complex curve ([2], see also [6] for an exposition and [5,7] for the relation to vertex algebras). The latter approach is based on representation of algebras in pseudotensor categories. For example, conformal algebras are algebras from the category of left $\mathbb{C}[\partial]$ -modules.

Using the language of pseudotensor categories, a natural generalization of conformal algebras was introduced in [3]. These objects, called pseudoalgebras, are also related to

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the differential Lie algebras of Ritt and Hamiltonian formalism in the theory of nonlinear evolution equations. In [3] a full classification of semisimple finite Lie pseudoalgebras was obtained, together with a foundation for their representation theory. One of the main objects there is the pseudoalgebra of pseudolinear operators of a finite module. Its study was the main inspiration for this paper.

Our ground field is \mathbb{C} ; the reader may replace it with the favorite algebraically closed field of zero characteristic.

1.1. Pseudoalgebras

A pseudoalgebra is a left module R over a Hopf algebra H together with an “operation” $R \otimes R \rightarrow (H \otimes H) \otimes_H R$ denoted $a * b$. Here the tensor product over H is defined via the comultiplication $\Delta: H \rightarrow H \otimes H$.

The Hopf structure allows to define the x -products $R \otimes R \rightarrow R$ denoted $a_x b$ for every x in the dual algebra $H^* = \text{Hom}(H, \mathbb{C})$. We have

$$a * b = \sum_i (S(h_i) \otimes 1) \otimes_H (a_{x_i} b), \quad \text{for dual bases } \{h_i\}, \{x_i\} \text{ of } H, H^*. \quad (1.1)$$

Thus, a pseudoalgebra can be defined as an H -module with a collection of bilinear operations indexed by H^* .

Consider the annihilation algebra of a pseudoalgebra R , $\mathcal{A}(R) = H^* \otimes_H R$. $\mathcal{A}(R)$ possesses a natural left H -action (on the first component of the tensor product). For $a, b \in R$, $a * b = \sum_i (f_i \otimes g_i) \otimes_H c_i$, multiplication is defined as

$$(x \otimes_H a)(y \otimes_H b) = \sum_i (x f_i)(y g_i) \otimes_H c_i. \quad (1.2)$$

When R is H -torsion free, the annihilation algebra $\mathcal{A}(R)$ almost completely describes R . In particular, R is a left $\mathcal{A}(R)$ -module: $(x \otimes_H a) \cdot b = a_x b$.

One can also define varieties of pseudoalgebras. In the Lie and associative cases, this requires extending the pseudoproducts to maps $(H^{\otimes 2} \otimes_H R) \otimes R \rightarrow H^{\otimes 3} \otimes_H R$ and $R \otimes (H^{\otimes 2} \otimes_H R) \rightarrow H^{\otimes 3} \otimes_H R$. This is done via the pseudotensor structure on the category of left H -modules. In particular, for $a, b \in R$, $a * b = \sum_i \beta_i \otimes_H c_i$, we have

$$\begin{aligned} (\gamma \otimes_H a) * b &= \sum_i (\gamma \otimes 1)(\Delta \otimes \text{id})(\beta_i) \otimes_H c_i, \\ a * (\gamma \otimes_H b) &= \sum_i (1 \otimes \gamma)(\text{id} \otimes \Delta)(\beta_i) \otimes_H c_i. \end{aligned}$$

Then associativity is given by a familiarly looking equality in $H^{\otimes 3} \otimes_H R$:

$$a * (b * c) = (a * b) * c.$$

Representations of pseudoalgebras (called modules) of a particular variety are defined in a similar way.

One of the most important examples of pseudoalgebras is Cend_n , a pseudoalgebra of pseudolinear operators of a finite H -module. As the category of H -modules plays a role of the category of vector spaces in the ordinary (non-pseudo)representation theory, Cend_n is an analogue of the algebra End_n . In particular, endowing a free H -module on n generators with a structure of an R -module for an associative pseudoalgebra R is the same as providing a map $R \rightarrow \text{Cend}_n$. As every associative pseudoalgebra can be made into a Lie one, a similar statement is true for Lie pseudoalgebras as well.

The most important examples of pseudoalgebras are those over a universal enveloping algebra $H = U(\mathfrak{g})$. We restrict ourselves to this case for the rest of the section; although, several constructions work for any cocommutative H .

1.2. Unital algebras and their representations

Any study of pseudoalgebras is ultimately a study of corresponding annihilation algebras. The standard trick in the study of ordinary associative algebras is to adjoin the identity; however, it is unclear if such an operation can be performed on the pseudoalgebra level, i.e., if we will still remain in the class of annihilation algebras. Thus, it is necessary to introduce some concept of “identity” for the pseudoalgebras themselves.

Identity in ordinary algebras comes from an embedding of the ground field \mathbb{C} . Similarly, for pseudoalgebras unitality is formulated as the existence of an embedding of the finite H -pseudoalgebra that is a free H -module of rk 1 such that its action on the whole pseudoalgebra is nonzero. Such pseudoalgebras are called *unital*. Of course, a pseudoalgebra of all pseudolinear endomorphisms of some H -module (in particular, Cend_n) is unital. The generator of the resulting subalgebra produces a left identity when passing to the annihilation algebra.

The structure of Cend_n can be described explicitly. Namely, $\text{Cend}_n = H \otimes H \otimes \text{End}_n(\mathbb{C})$ and the multiplication is completely determined by that in the ordinary algebra $H \otimes \text{End}_n(\mathbb{C})$ and the H^* -action on it. More generally, let A be an $(H^*)^{\text{cop}}$ -differential algebra (when H is commutative, this simply means that elements of H^* acts on A as differential operators, see (2.5)) such that the action of the augmentation ideal of H^* is locally nilpotent. Then A gives rise to a pseudoalgebra $\text{Diff } A = H \otimes A$ where the multiplication is defined via (1.1) with $(1 \otimes a)_{x_i}(1 \otimes b) = 1 \otimes (a \cdot x_i(b))$ for $a, b \in A$. Such pseudoalgebras are called *differential* (see Example 3.8). They can be easily made unital (by adjoining identity to A). In most cases the converse is also true.

Theorem 1.1. *A semisimple unital associative pseudoalgebra is differential.*

Passing from $\text{Diff } A$ to A (whose structure is easier to understand than that of the annihilation algebra) allows for the study and, in some sense, complete classification of representations of $\text{Diff } A$.

Theorem 1.2. *Let V be a representation of unital differential pseudoalgebra $R = \text{Diff } A$. Then $V = V^0 \oplus V^1$, where $R * V^0 = 0$ and V^1 is constructed from a unitary*

A -module. Moreover, V^1 is irreducible (indecomposable) if and only if A is irreducible (indecomposable).

In particular, the above theorem allows to classify representations of Cend_n .

1.3. Classification results

Let us now try to answer the question what algebras are “similar” to Cend_n (i.e., have a similar algebraic structure). Clearly, the necessary conditions must be simplicity and unitality. A finiteness condition must be a requirement as well; even though, Cend_n is not finite over H . In [15] we classified conformal algebras of linear growth; however, this is not a good condition for the case of a general H .

Given a pseudoalgebra R over a Hopf subalgebra H' of H , it is easy to lift it to an H -pseudoalgebra together with its modules; this construction is called the current extension $\text{Cur}_{H'}^H R$ (see Example 3.1). Therefore, it is only natural to include in any classification current extensions of H' -pseudoalgebras with the same properties. On the other hand, the growth of such current extensions does not change, so it is possible to produce a current H -pseudoalgebra that is “small” compared to H but has properties of the original algebra. Hence, we ought to rely not on comparison of growth but on some absolute criterion which does not change with passing to the current extension. Since in the ordinary ($H = \mathbb{C}$) case the algebras $\text{Cend}_n = \text{End}_n(\mathbb{C})$ are finite, we shall require a presence of a certain finite current subalgebra.

As we remarked above, Cend_n is an example of a differential pseudoalgebra. The multiplication in Cend_n is characterized by the action of H^* on $H \otimes \text{End}_n(\mathbb{C})$ by differential operators. This action survives passing to the universal enveloping algebra of a central extension of \mathfrak{g} . In particular, let $0 \rightarrow \mathbb{C}c \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 1$ be determined by a cocycle ϕ . H^* naturally acts on $U(\hat{\mathfrak{g}})/(1-c)$ and, consequently, on its tensor product with $\text{End}_n(\mathbb{C})$ (for details see Example 3.12). This defines the differential pseudoalgebra Cend_n^ϕ .

Theorem 1.3. *Let R be a simple unital associative H -pseudoalgebra. Assume that its maximal unital current subalgebra having the same pseudoidentity as R is finite as an H -module and simple. Then R is either of*

- $\text{CurEnd}_n(\mathbb{C})$, $n > 0$;
- $\text{Cur}_{H'}^H \text{Cend}_n$, H' a Hopf subalgebra of H , $n > 0$;
- $\text{Cur}_{H'}^H \text{Cend}_n^\phi$, $H' = U(\mathfrak{h})$ a Hopf subalgebra of H , $\phi \in H^2(\mathfrak{h}, \mathbb{C})$, $n > 0$.

Remark that R above is automatically finitely generated.

Even for a commutative H , Cend_n and Cend_n^ϕ (which is constructed from the Weyl algebra) have very different representations. However, their algebraic properties are remarkably similar. This discrepancy, of course, does not occur in the ordinary ($H = \mathbb{C}$) and the conformal ($H = \mathbb{C}[\partial]$) cases.

Remark 1.4. I am unaware of another algebraic classification where objects in a given class are parametrized by $\bigcup_{\mathfrak{h} \subset \mathfrak{g}} H^2(\mathfrak{h}, \mathbb{C})$ for a finite-dimensional Lie algebra \mathfrak{g} . For an

arbitrary finite group G , elements of $\bigcup_{H \leq G} H^2(H, \mathbb{C})$ correspond to indecomposable modular categories over $\text{Rep } G$ [14] but this result can not be carried over to the case of $\text{Rep } g$.

1.4. Organization of the paper

We start by defining in Section 2 the main objects of our study: pseudoalgebras and their representations. We also recall some useful facts about Hopf algebras. This section mainly follows a similar discussion in the first chapters of [3].

Section 3 is devoted to examples of pseudoalgebras. In particular, we will define the differential pseudoalgebras in Example 3.8 and discuss in greater detail the structure of Cend_n .

In Section 4 we define unital pseudoalgebras and show in Theorem 4.9 that under a certain technical condition such algebras are differential over unital algebras. Theorem 1.1 follows (Corollary 4.11). The proof itself expands that of Proposition 3.5 in [15]. We also briefly touch upon the classification of unital algebras over cocommutative Hopf algebras.

In Section 5 we describe representations of unital differential pseudoalgebras. In particular, we explicitly show how to construct a unitary module of a differential pseudoalgebra $\text{Diff } A$ from a unitary A -module (Lemma 5.5). Theorem 1.2 follows. As a corollary we derive the classification of indecomposable modules of the conformal algebra Cend_n . It was originally stated in [11]; however, our methods are different and do not refer to the Lie case. A recent paper [1] also contains this result (for more on this approach, see Remark 5.16).

Finally, in Section 6 we prove Theorem 1.3. This essentially boils down to classifying certain filtered algebras with a locally nilpotent X^{cop} -action. When X is cocommutative, these algebras are differentially simple; however, our methods are different from other studies of differentially simple algebras, as there are no minimality conditions (see [4] and references therein, and also [12]). We also discuss in greater detail the relation between properties of pseudoalgebras and their growth and conjecture a stronger version of Theorem 1.3.

2. A short survey of pseudoalgebras

2.1. Pseudotensor categories

Theory of pseudotensor categories was developed in [2] as a way of expressing such notions as Lie algebras, representations etc. in purely categorical terms. The ultimate goal is to define these notions for categories of modules that have an interesting action on tensor products, e.g., \mathcal{D} -modules (as in [2]) or modules over a Hopf algebra (as in [3] or this paper).

More details on pseudotensor categories can be found in [2, Chapter 1] and [3, Chapter 3], we will get by with a short presentation of main definitions and several examples.

Denote by \mathcal{S} the category of finite non-empty sets with surjective maps. For a morphism $\pi : J \twoheadrightarrow I$ and $i \in I$, we denote $\pi^{-1}(i) = J_i$.

Definition 2.1. A pseudotensor category is a class of objects \mathcal{M} together with the following data:

- For any $I \in \mathcal{S}$, a family of objects $\{L_i\}_{i \in I}$ and an object M , one has the set of *polylinear maps* $\text{Lin}_I(\{L_i\}, M)$; the symmetric group S_I acts on such $\text{Lin}_I(\{L_i\}, M)$ by permuting the factors.
- For any morphism $\pi : J \twoheadrightarrow I$ in \mathcal{S} , the families of objects $\{L_i\}_{i \in I}$ and $\{N_j\}_{j \in J}$ and an object M , there exists the composition map

$$\text{Lin}_I(\{L_i\}, M) \otimes \bigotimes_{i \in I} \text{Lin}_{J_i}(\{N_j\}, L_i) \rightarrow \text{Lin}_J(\{N_j\}, M) \ni \phi(\{\psi_i\}_{i \in I}).$$

This data satisfies the following properties:

Associativity: For a surjective map $K \twoheadrightarrow J$ and a K -family of objects $\{P_k\}_{k \in K}$, one has $\phi(\{\psi_i(\{\chi_j\})\}) = (\phi(\{\psi_i\}))(\{\chi_j\}) \in \text{Lin}_K(\{P_k\}, M)$, given $\chi_j \in \text{Lin}_{K_j}(\{P_k\}, N_j)$;

Unit: For any object M , there exists an element $\text{id}_M \in \text{Lin}(\{M\}, M)$ such that for any $\phi \in \text{Lin}_I(\{L_i\}, M)$, one has $\text{id}_M(\phi) = \phi(\{\text{id}_{L_i}\}) = \phi$;

Equivariance: The compositions of polylinear maps are equivariant with respect to the natural action of the symmetric group.

Examples 2.2. (1) For the category $\mathcal{V}ec$ of vector spaces, put

$$\text{Lin}_I(\{L_i\}, M) = \text{Hom}(\bigotimes_i L_i, M).$$

The symmetric group acts on $\text{Lin}_I(\{L_i\}, M)$ by permuting the factors of $\bigotimes_i L_i$.

(2) Let H be a cocommutative bialgebra with a comultiplication $\Delta : H \rightarrow H^{\otimes 2}$ and $\mathcal{M}^l(H)$ its category of left modules. This is a symmetric tensor category; hence, it can be made into a pseudotensor category: $\text{Lin}_I(\{L_i\}, M) = \text{Hom}_H(\bigotimes_i L_i, M)$.

(3) We will introduce another pseudotensor structure on $\mathcal{M}^l(H)$.

Recall that Δ gives rise to a functor $\mathcal{M}^l(H) \rightarrow \mathcal{M}^l(H^{\otimes 2})$, $M \mapsto H^{\otimes 2} \otimes_H M$, where H acts on $H^{\otimes 2}$ via Δ . This may be generalized as follows. For every surjection $\pi : J \twoheadrightarrow I$, define a functor $\Delta^{(\pi)} : \mathcal{M}^l(H^{\otimes I}) \rightarrow \mathcal{M}^l(H^{\otimes J})$, $M \mapsto H^{\otimes J} \otimes_{H^{\otimes I}} M$, where $H^{\otimes I}$ acts on $H^{\otimes J}$ via the iterated comultiplication determined by π (the i th copy of H is mapped into $H^{\otimes J_i}$). This is well-defined because of coassociativity.

Denote the tensor product functor $\mathcal{M}^l(H)^I \rightarrow \mathcal{M}^l(H^{\otimes I})$ by $\boxtimes_{i \in I}$. Then we can define a pseudotensor category $\mathcal{M}^*(H)$ that has the same objects as $\mathcal{M}^l(H)$ but with

$$\text{Lin}_I(\{L_i\}, M) = \text{Hom}_{H^{\otimes I}}(\boxtimes_{i \in I} L_i, H^{\otimes I} \otimes_H M). \quad (2.1)$$

For $\pi : J \twoheadrightarrow I$, the composition of polylinear maps is defined as follows:

$$\phi(\{\psi_i\}) = \Delta^{(\pi)}(\phi) \circ (\boxtimes_i \psi_i). \quad (2.2)$$

The symmetric group acts on $\text{Lin}_I(\{L_i\}, M)$ by simultaneously permuting the factors in $\boxtimes_i L_i$ and $H^{\otimes I}$. This is well-defined because of cocommutativity.

Examples of explicit calculations in $\mathcal{M}^*(H)$ will be provided below.

A *Lie algebra* in a pseudotensor category \mathcal{M} is an object L together with a polylinear map $\beta \in \text{Lin}(\{L, L\}, L)$ that satisfies analogues of skew-commutativity and the Jacobi identity: $\beta = -(12)\beta$, where $(12) \in S_2$, and $\beta(\beta(\cdot, \cdot), \cdot) = \beta(\cdot, \beta(\cdot, \cdot)) - (12)\beta(\cdot, \beta(\cdot, \cdot))$, where (12) now lies in S_3 .

An *associative algebra* is an object $R \in \mathcal{M}$ together with a polylinear map $\mu \in \text{Lin}(\{R, R\}, R)$ satisfying associativity $\mu(\mu(\cdot, \cdot), \cdot) = \mu(\cdot, \mu(\cdot, \cdot))$.

In \mathcal{Vec} associative (Lie) algebras are just the associative (Lie) algebras in their usual sense. To avoid confusion we will sometimes call these algebras *ordinary*. The same is true of representations (to be defined below), cohomology (see [3]), etc.

Remark 2.3. Consider an associative algebra (R, μ) in a pseudotensor category \mathcal{M} . Then the pair $(R, \mu - (12)\mu)$ is a Lie algebra in \mathcal{M} [3, Proposition 3.1]. However, not every Lie algebra can be represented in this form, i.e., the PBW theorem does not necessarily hold in a generic pseudotensor category (see [17] for the case of conformal algebras).

A *representation* of an associative algebra (R, μ) in \mathcal{M} is an object M (a *module*) together with $\rho \in \text{Lin}(\{R, M\}, R)$ satisfying $\rho(\mu(\cdot, \cdot), \cdot) = \rho(\cdot, \rho(\cdot, \cdot))$. Representations of Lie algebras are defined in the same way.

2.2. Preliminaries on Hopf algebras

Before proceeding further, we to recall several facts about Hopf algebras (see [9, Chapter 1] or [18] for basic definitions and notations).

In this paper H will always stand for a Hopf algebra with a coproduct Δ , a counit ε , and an antipode S . As usual, we use Sweedler's notations: $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (summation is implied), $(\Delta \otimes \text{id})\Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$, $(S \otimes \text{id})\Delta(h) = h_{(-1)} \otimes h_{(2)}$ etc.

The following formulas will be quite useful:

$$\varepsilon(h_{(1)})h_{(2)} = h_{(1)}\varepsilon(h_{(2)}) = h, \quad (2.3)$$

$$h_{(-1)}h_{(2)} \otimes h_{(3)} = h_{(1)}h_{(-2)} \otimes h_{(3)} = 1 \otimes h. \quad (2.4)$$

H also possesses the opposite coproduct $\Delta^{\text{cop}}: h \mapsto h_{(2)} \otimes h_{(1)}$; we denote the Hopf algebra $(H, \Delta^{\text{cop}}, \varepsilon, S)$ as H^{cop} . As usual, H^{op} will stand for the algebra H with the opposite multiplication.

An associative algebra A is called *H-differential* if it is a left H -module such that

$$h(xy) = (h_{(1)}x)(h_{(2)}y). \quad (2.5)$$

Remark 2.4. An *H-differential algebra* is an associative algebra in the pseudotensor category $\mathcal{M}^l(H)$.

For an H -differential algebra A one defines a *smash product* $A \sharp H$ as a tensor product $A \otimes H$ of underlying vector spaces with a new multiplication

$$(a \sharp g)(b \sharp h) = a(g_{(1)}b) \sharp g_{(2)}h.$$

$G(H)$ stands for the set of group-like elements of H , i.e., $h \in H$ such that $\Delta(h) = (h \otimes h)$. $P(H)$ is the set of primitive elements, i.e., $h \in H$ with $\Delta(h) = 1 \otimes h + h \otimes 1$. Group-like elements form a group with multiplication inherited from H , and $P(H)$ is a Lie subalgebra of H with respect to the standard commutator $[g, h] = gh - hg$.

A typical example of a smash product is $U(P(H)) \sharp \mathbb{C}[G(H)]$, where $G(H)$ acts on $P(H)$ by inner automorphisms: $g(h) = gp g^{-1} \in P(H)$.

Remark 2.5. A theorem due to Kostant [18, Theorem 8.1.5] states that a cocommutative Hopf algebra H is, in fact, isomorphic to $U(P(H)) \sharp \mathbb{C}[G(H)]$.

We will also require a standard filtration on H : $F^0 H = \mathbb{C}[G(H)]$ and for $n > 0$,

$$F^n H = \left\{ h \in H \mid \Delta(h) \in F^0 H \otimes h + h \otimes F^0 H + \sum_{i=1}^{n-1} F^i H \otimes F^{n-i} H \right\}.$$

When $H = U(\mathfrak{g})$ is a universal enveloping algebra, we get the canonical filtration. Remark that when \mathfrak{g} is finite-dimensional, $\dim F^n H < \infty$ for all n . It is clear that operations on H respect the filtration.

When H is cocommutative, Remark 2.5 implies $\bigcup_n F^n H = H$. If so, we say that a nonzero element $h \in H$ has *degree* n if $h \in F^n H \setminus F^{n-1} H$.

In order to define certain operations on pseudoalgebras (see below), we will need the following:

Lemma 2.6 [3, Lemma 2.3]. *Every element of $H \otimes H$ can be uniquely represented in the form $\sum_i (h_i \otimes 1) \Delta(l_i)$, where $\{h_i\}$ is a fixed basis of H and $l_i \in H$. Also, for any H -module V ,*

$$(F^n H \otimes \mathbb{C}) \otimes_H V = F^n (H \otimes H) \otimes_H V = (\mathbb{C} \otimes F^n H) \otimes_H V,$$

where $F^n (H \otimes H) = \sum_{i+j=n} F^i H \otimes F^j H$.

2.3. Dual algebra of a Hopf algebra

Denote the dual algebra of H by $X = H^* = \text{Hom}_{\mathbb{C}}(H, \mathbb{C})$.

It is an H -differential algebra with the action defined by

$$\langle hx, f \rangle = \langle x, S(h)f \rangle, \quad \text{for } f, h \in H, x \in X.$$

One can similarly define the structure of a right H -module on X ; this makes X into an H -bimodule.

X possesses a standard filtration $X = F_{-1}X \supset F_0X \supset \cdots$, where $F_nX = (F^n H)^\perp$.

When H is cocommutative, X is commutative and $\bigcap_n F_nX = 0$.

By a basis of X we will always mean a topological basis $\{x_i\}$ such that for any n only a finite number of x_i 's does not lie in F_nX (i.e., $x_i \rightarrow 0$ in the standard topology). Let $\{h_i\}$ be a basis of H compatible with the standard filtration. If $\dim F^n H < \infty$ for all n , the dual basis of X is topological. For $h \in H$ and $x \in X$, we have

$$h = \sum_i \langle h, x_i \rangle h_i, \quad x = \sum_i \langle x, h_i \rangle x_i,$$

where the first sum is finite and the second converges in the standard topology.

Define the antipode S on X as a dual of that of H : $\langle S(x), h \rangle = \langle x, S(h) \rangle$. Also, we introduce a coproduct $\Delta: X \rightarrow X \hat{\otimes} X$ where $X \hat{\otimes} X = (H \otimes H)^*$ is the completed tensor product. By definition, for $x, y \in X$ and $f, g \in H$

$$\langle xy, f \rangle = \langle x \otimes y, \Delta(f) \rangle = \langle x, f_{(1)} \rangle \langle y, f_{(2)} \rangle, \quad (2.6)$$

$$\langle x, fg \rangle = \langle \Delta(x), f \otimes g \rangle = \langle x_{(1)}, f \rangle \langle x_{(2)}, g \rangle. \quad (2.7)$$

Remark 2.7. For X such that $\dim F_nX < \infty$ for all n , one can endow H with the structure of an X -differential algebra. As in the case of H -action on X , the action is defined by $\langle y, xh \rangle = \langle S(x)y, h \rangle$. The proof is also similar to that for the H -action on X and uses (2.7) instead of (2.6) and (2.4). Formula (2.5) makes sense as the right-hand side will be finite for every pair of elements of H .

2.4. Notations for universal enveloping algebras

In this paper we mostly restrict our attention to universal enveloping algebras of finite-dimensional Lie algebras. Some notations are in order:

Put $H = U(\mathfrak{g})$ where \mathfrak{g} is a n -dimensional Lie algebra spanned over \mathbb{C} by $\{\partial_i\}_{i=1}^n$. We fix the canonical (but not PBW) basis of H indexed by elements of $\mathbb{Z}_{\geq 0}^n$:

$$\partial^I = \frac{\partial_1^{i_1} \cdots \partial_n^{i_n}}{i_1! \cdots i_n!} \quad \text{for } I = (i_1, \dots, i_n).$$

It is easy to see that $\Delta(\partial^I) = \sum_{J+K=I} \partial^J \otimes \partial^K$.

Remark 2.8. For future reference, we need to describe our notations for the multiindex set $\mathbb{Z}_{\geq 0}^n$. The addition is pointwise. There is the standard partial ordering, i.e., $(i_1, \dots, i_n) > (j_1, \dots, j_n)$ iff $i_m > j_m$ for all m ; if neither $I \geq J$ nor $J \geq I$, we call I and J *incompatible*. The index $(0, \dots, 0)$ is denoted simply by 0. Also, for a multiindex $I = (i_1, \dots, i_n)$ we put $|I| = i_1 + \cdots + i_n$ and $(-1)^I = (-1)^{|I|}$.

The dual Hopf algebra of H is $X = \mathbb{C}[[t_1, \dots, t_n]]$ with the canonical dual basis $t^I = t_1^{i_1} \cdots t_n^{i_n}$. As usual, $t^0 = 1$. The action of H on X is given by differential operators: $\partial_i = -\partial/\partial t_i + \text{higher order terms}$. The action of X on H is defined similarly.

Remark 2.9. The standard filtration on X regarded as the dual algebra of H is not the standard decreasing filtration on the polynomial algebra $\mathbb{C}[t_1, \dots, t_n]$ (it is shifted by 1). In particular, the standard total degree function on X does not respect multiplication.

It is easy to see that $\Delta(t_i) = 1 \otimes t_i + t_i \otimes 1 + \text{summands with both sides of degree higher than 0}$. This can be generalized for any t^I :

$$\Delta(t^I) = \sum_{J \leq I} t^J \otimes t^{I-J} + \sum_j c_j t^{K_j} \otimes t^{L_j}, \quad |K_j| + |L_j| \geq |I| + 1, \quad c_j \in \mathbb{C}. \quad (2.8)$$

In particular, one can deduce from (2.8) that

$$\Delta(F_{n-1}X) \subset \sum_{i=0}^n F_{i-1}X \hat{\otimes} F_{n-i-1}X. \quad (2.9)$$

2.5. Pseudoalgebras and their representations

Recall the description of the pseudotensor category $\mathcal{M}^*(H)$ (see Example 2.2(3)).

Definition 2.10. An associative (Lie) *pseudoalgebra* over a cocommutative Hopf algebra H is an associative (Lie) algebra in $\mathcal{M}^*(H)$. A representation of a pseudoalgebra (a *module*) is its representation in $\mathcal{M}^*(H)$.

We denote multiplication in a pseudoalgebra R by $*$: $R \otimes R \rightarrow (H \otimes H) \otimes_H R$ and call $a * b$ the *pseudoproduct* of a and b .

This operation satisfies H -bilinearity: for $f, g \in H$,

$$(fa) * (gb) = ((f \otimes g) \otimes_H 1)(a * b),$$

and associativity $(a * b) * c = a * (b * c)$. The explicit expressions for the latter equality are calculated below, following [3, Chapter 3].

Let $a, b, c \in R$. To calculate $(a * b) * c \in H^{\otimes 3} \otimes_H R$ in accordance with (2.2), notice that here $J = \{1, 2, 3\}$, $I = \{1, 2\}$, $\psi_1 = \phi = *$, $\psi_2 = \text{id}$, and the map $\pi: J \rightarrow I$ is given by $\pi(1) = \pi(2) = 1$, $\pi(3) = 2$. Put

$$a * b = \sum_i (f_i \otimes g_i) \otimes_H d_i,$$

$$d_i * c = \sum_j (f_{ij} \otimes g_{ij}) \otimes_H d_{ij}.$$

Then, as $\Delta^{(\pi)} = \Delta \otimes \text{id}$,

$$(a * b) * c = \sum_{i,j} (f_i f_{ij(1)} \otimes g_i f_{ij(2)} \otimes g_{ij}) \otimes_H d_{ij}. \quad (2.10)$$

Similarly, for the product $a * (b * c)$, $\Delta^{(\pi)} = \text{id} \otimes \Delta$, and we obtain

$$a * (b * c) = \sum_{ij} (h_{ij} \otimes h_i k_{ij(1)} \otimes k_j k_{ij(2)}) \otimes_H e_{ij},$$

$$\text{where } b * c = \sum_i (h_i \otimes k_i) \otimes_h e_i, \quad a * e_i = \sum_j (h_{ij} \otimes k_{ij}) \otimes_h e_{ij}. \quad (2.11)$$

For a representation V of R , we will also denote the action by $*$: $a * v \in (H \otimes H) \otimes_H V$. It also satisfies H -bilinearity and associativity.

Remark 2.11. Recall that a cocommutative Hopf algebra is a smash product of $G(H)$ and $U(P(H))$ (Remark 2.5). For brevity denote $\Gamma = G(H)$, $H' = U(P(H))$. The action of Γ on H' can be extended to the action on $(H')^{\otimes I}$ via $\Delta^{(I)}(g) = \otimes_i g$, and in an obvious way to the action on $(H')^{\otimes I} \otimes_{H'} M$ for an H' -module M . It can be shown that the category $\mathcal{M}^*(H)$ is equivalent to a subcategory of $\mathcal{M}^*(H')$ that consists of H -modules and polylinear maps that commute with the action of Γ . Thus, it follows [3, Corollary 5.1] that an H -pseudoalgebra is an H' -pseudoalgebra with an action of Γ such that $ga * gb = g(a * b)$. Moreover, the pseudoproduct over H is defined as

$$a * b = \sum_{g \in \Gamma} ((g^{-1} \otimes 1) \otimes_H 1)(ga * b), \quad (2.12)$$

where the products on the RHS are taken over H' and the sum is finite.

This shows that the case of pseudoalgebras over general cocommutative algebras in most cases reduces to the study of pseudoalgebras over universal enveloping algebras.

2.6. Annihilation algebra and x -products

As before, $X = H^*$. Let R be a left module of a cocommutative Hopf algebra H . Define another H -module $\mathcal{A}(R) = X \otimes_H R$ with an obvious left action $h(x \otimes_H a) = hx \otimes_H a$. If R is also an associative H -pseudoalgebra, $\mathcal{A}(R)$ is an associative algebra with multiplication defined by (1.2) [3, Proposition 7.1].

$\mathcal{A}(R)$ is an H -differential algebra. Remark also that an R -module V gives rise to an $\mathcal{A}(R)$ -module $\mathcal{A}(V) = X \otimes_H V$.

The algebra $\mathcal{A}(R)$ is called the *annihilation algebra* of the pseudoalgebra R . Its elements $x \otimes_H a$ are denoted as a_x and are called *Fourier coefficients* of a .

The annihilation algebra closely mirrors the properties of the corresponding pseudoalgebra. In particular, when R is torsion-free, $\mathcal{A}(R)$ “distinguishes” its elements:

Lemma 2.12 (cf. [3, Proposition 11.3]). *Let M be a left H -module over a universal enveloping algebra H . All Fourier coefficients of $a \in M$ are zero if and only if a is torsion.*

Moreover, sometimes it is possible to get back from $\mathcal{A}(M)$ to M . Given a topological left H -module L , one can construct another module $\mathcal{C}(L) = \text{Hom}_H^{\text{cont}}(X, L)$ (“cont” stands

for continuous in the standard topologies of X and H). Define the map $\Phi : M \rightarrow \mathcal{C}(\mathcal{A}(M))$ as $\Phi(a)(x) = x \otimes_H a$. In most interesting cases, M imbeds into $\Phi(M)$ and, if M possesses a pseudoalgebra structure, so does $\Phi(M)$ (see Lemma 2.13).

Let R be an associative pseudoalgebra with the pseudoproduct

$$a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i.$$

The choice of f_i , g_i , and e_i is certainly not unique. By Lemma 2.6 we can assume that $g_i = 1$. This defines the new operation $R \otimes R \rightarrow H \otimes R$: $a \cdot b = \sum_i f_i \otimes e_i$. For any $x \in X$ we introduce the x -product (see [3, Chapter 9]):

$$a_x b = (\langle S(x), \cdot \rangle \otimes \text{id}) a \cdot b = \sum_i \langle S(x), f_i \rangle e_i. \quad (2.13)$$

Given the x -products of a and b , one can also pass back to their pseudoproduct, obtaining (1.1).

Notice that the sum in (1.1) is finite, i.e., for almost all x_i , $a_{x_i} b = 0$.

Thus, one can define an associative H -pseudoalgebra as a left H -module R equipped with the x -products satisfying:

Locality:

$$\text{codim}\{x \in X \mid a_x b = 0\} < \infty \quad \text{for any } a, b \in R; \quad (2.14)$$

H -sesquilinearity:

$$\begin{aligned} (ha)_x b &= a_{xh} b, \\ a_x(hb) &= h_{(2)}(a_{h_{(-1)x}} b) \quad \text{for any } a, b \in R, h \in H; \end{aligned} \quad (2.15)$$

Associativity:

$$a_x(b_y c) = (a_{x_{(2)}} b)_{x_{(1)}y} c. \quad (2.16)$$

Locality suggests the following definition: we will call $x \in X$ *maximal with respect to* a and b if $a_x b \neq 0$ but for any $y \in F_0 X$, $a_{xy} b = 0$.

Associativity (2.16) can be equivalently stated as

$$(a_x b)_y c = a_{x_{(2)}}(b_{x_{(-1)}y} c). \quad (2.17)$$

Most of the above properties survive the passing to $\mathcal{C}(\mathcal{A}(R))$:

Lemma 2.13 (cf. [3, Proposition 11.1]). *Let R be an associative pseudoalgebra. Then $\Phi(R) = \mathcal{C}(\mathcal{A}(R))$ satisfies (2.15)–(2.17).*

The above formulas and statements, of course, remain true for the action of R on its module M and of $\mathcal{A}(R)$ on $\mathcal{A}(M)$.

Using (1.1) we can obtain formulas similar to (2.16) and (2.17) for the multiplication in $\mathcal{A}(R)$:

$$\begin{aligned} a_x \cdot b_y &= (a_{x(2)}b)_{x(1)y}, \\ (a_xb)_y &= (a_{x(2)}) \cdot (b_{x(-1)y}). \end{aligned} \quad (2.18)$$

We can now define structural concepts for associative pseudoalgebras. Denote by A_xB the set $\{a_xb \mid a \in A, b \in B\}$. An *ideal* I of R is a pseudoalgebra such that for all $x \in X$, $I_xR \subset I$, $R_xI \subset I$. A pseudoalgebra R whose only ideals are 0 and R is called *simple*. A pseudoalgebra R such that for a fixed n , $R_{x_1}R_{x_2} \cdots R_{x_n}R = 0$ for any collection of $\{x_1, \dots, x_n\} \subset X$ is called *nilpotent* (as x_i 's are arbitrary, we can omit the brackets). A pseudoalgebra that contains no nilpotent ideals is called *semisimple*. Similar definitions for modules will be provided in Section 5.

3. Examples of associative pseudoalgebras

In this section we provide several important examples of associative pseudoalgebras. In general, we do not assume that H is a universal enveloping algebra of a Lie algebra.

3.1. General examples

Example 3.1. Let H' be a Hopf subalgebra of H , and R an H' -pseudoalgebra.

Definition 3.2. The *current extension* of R is the H -pseudoalgebra $\text{Cur}_{H'}^H R$ which is the H -module $H \otimes_{H'} R$ with the pseudoproduct $*$ extending the pseudoproduct of R by H -bilinearity.

More explicitly, for $a, b \in R$ with $a * b = \sum_i (f_i \otimes g_i) \otimes_{H'} c_i$ we define

$$\begin{aligned} (f \otimes_{H'} a) * (g \otimes_{H'} b) &= ((f \otimes g) \otimes_H 1)(a * b) \\ &= \sum_i ((ff_i) \otimes (gg_i)) \otimes_H (1 \otimes_{H'} c_i). \end{aligned} \quad (3.1)$$

The same construction, of course, is possible for any variety of (pseudo)algebras defined over H .

Remark 3.3. This terminology is different from that of [3] where current extensions were called current pseudoalgebras. However, here this term is restricted to a smaller class (see immediately below). In the author's view this makes some statements below less cluttered.

In particular, when $H' = \mathbb{C}$, an associative H' -pseudoalgebra R is an associative \mathbb{C} -algebra with the ordinary product. Then $\text{Cur}_{H'}^H R$, which we will denote simply $\text{Cur } R$, has the pseudoproduct

$$(f \otimes a) * (g \otimes b) = (f \otimes g) \otimes_H (1 \otimes ab).$$

We will call such a pseudoalgebra $\text{Cur } R$ a *current pseudoalgebra*.

Current pseudoalgebras have a simple characterization: if in a pseudoalgebra R $a_x b = 0$ for all $a, b \in R$ and $x \in F_0 X$, then R is current.

Example 3.4. Let $H = U(\mathfrak{g})$ be a universal enveloping algebra and R an associative pseudoalgebra over H that is free and of rk 1 as an H -module. Below we classify such pseudoalgebras.

Lemma 3.5. *Let R be as above. Then either the multiplication in R is trivial (i.e., $a * b = 0$ for any $a, b \in R$) or $R \cong \text{Cur } \mathbb{C}$.*

Proof. Let e be a generator of R over H , i.e., $R = He$. By H -bilinearity, multiplication in R is completely determined by the values of the coefficients in the product $e * e$. Namely, put

$$e * e = \alpha \otimes_H e, \quad \text{where } \alpha = \sum_{(I,J)} c_{IJ} \partial^I \otimes \partial^J \in H \otimes H.$$

Then, to classify pseudoalgebras of rk 1, it suffices to classify all appropriate α 's.

Associativity implies

$$(\alpha \otimes 1)(\Delta \otimes \text{id})(\alpha) = (1 \otimes \alpha)(\text{id} \otimes \Delta)(\alpha),$$

which can be rewritten as

$$\begin{aligned} \sum_{(I',J'),(I,J),K+L=I} c_{IJ} c_{I'J'} \partial^{I'} \partial^K \otimes \partial^{J'} \partial^L \otimes \partial^J \\ = \sum_{(I',J'),(I,J),M+N=J} c_{IJ} c_{I'J'} \partial^I \otimes \partial^{I'} \partial^M \otimes \partial^{J'} \partial^N. \end{aligned} \quad (3.2)$$

Pick I' with a maximal degree among all such that $c_{I'J'} \neq 0$. Then by comparing the degrees of the first terms in (3.2), we see that all $|K| = 0$, so $|I| = 0$. Similarly, $|J| = 0$, i.e., $\alpha = c \otimes 1$ for some $c \in \mathbb{C}$. For a nonzero c we can normalize e , so that $e * e = (1 \otimes 1) \otimes_H e$. This makes R isomorphic to $\text{Cur } \mathbb{C}$. \square

Lie pseudoalgebras of rk 1 were classified in [3, 4.3] by essentially similar methods.

3.2. Conformal algebras

Example 3.6. Below we shall partly follow the introduction to [3].

Let $H = \mathbb{C}[\partial]$ (and $X = \mathbb{C}[[t]]$). Then H -pseudoalgebras (of any variety) are conformal algebras [10, Chapter 2]. A non-axiomatic description of such objects is possible.

Namely, for an arbitrary ordinary algebra, consider the algebra $A[[z, z^{-1}]]$ whose elements are called formal distributions. We introduce bilinear products $f(z)g(z)$, $n \in \mathbb{Z}_{\geq 0}$ defined as

$$f(z)g(z)_{(n)} = \text{Res}_{w=0} f(w)g(z)(w-z)^n,$$

where $\text{Res}_{w=0} : A[[w, z, w^{-1}, z^{-1}]] \rightarrow A[[z, z^{-1}]]$ maps a formal distribution $h(w, z)$ in two variables to the coefficient at w^{-1} , and the product of two formal distributions in different variables is defined in the usual way. Under certain conditions, $f(w)g(z)$ can be expressed via the products $f(z)g(z)_{(n)}$; this is called the operator product expansion [10, Proposition 2.2].

There is a natural action of $\partial = \partial_z$ on $A[[z, z^{-1}]]$. We call an algebra of formal distributions *conformal* if it satisfies the locality property: for any f, g only a finite number of products $f(z)g(z)_{(n)}$ is nonzero.

Every pseudoalgebra R over H can be described in this way for $A = X[[t^{-1}]] \otimes_H R$. Operations are related as $f(z)g(z)_{(n)} = f_t^n g$.

The above construction obviously generalizes for the case of abelian \mathfrak{g} .

3.3. Pseudolinear algebras

Let V, W be H -modules. An H -pseudolinear map from V to W is a linear map $\phi : V \rightarrow (H \otimes H) \otimes_H W$ such that $\phi(hv) = ((1 \otimes h) \otimes 1)\phi(v)$ for $h \in H, v \in V$. The space of all such maps, denoted $\text{Chom}(V, W)$, is a left H -module: put $(h\phi)(v) = ((h \otimes 1) \otimes_H 1)\phi(v)$. When $V = W$, we denote the set of all pseudolinear maps as $\text{Cend}(V)$. Though it is possible to define the action of the product $\phi * \psi$ on V for $\phi, \psi \in \text{Cend } V$, it might not be represented by a finite sum, i.e., $\text{Cend } V$ is not necessarily a pseudoalgebra. However, when V is finite over H , $\text{Cend } V$ becomes an associative H -pseudoalgebra with a naturally defined multiplication.

Example 3.7. If V is a finite free H -module, i.e., $V = H \otimes V_0$ for some finite-dimensional vector space V_0 with a trivial action of H , then $\text{Cend } V = H \otimes H \otimes \text{End } V_0$ with the pseudoproduct defined as

$$(f \otimes a \otimes A) * (g \otimes b \otimes B) = (f \otimes ga_{(1)}) \otimes_H (1 \otimes ba_{(2)} \otimes AB) \quad (3.3)$$

(see [3, Propositions 10.1, 10.3]).

Clearly, in the above case $\text{Cend } V$ depends only on $\text{rk } V$. To emphasize this, for a module of rank n its pseudoalgebra of endomorphisms will be denoted simply Cend_n .

It is not difficult to see that Cend_n is simple [3, Proposition 13.12]; however, unlike the case of ordinary algebras of linear endomorphisms, it is not finite as an H -module.

When $H = \mathbb{C}[\partial]$, the conformal algebra Cend_n is sometimes called the conformal Weyl algebra \mathfrak{W}_n [15]. In this case the standard model of a finite H -module of $\text{rk } n$ is the module $E_n = \{a(z) = \sum_{a \in \mathbb{C}^n} at^n z^{-n-1}\}$. Here the generators of Cend_n over H are elements $J_A^m = \sum At^n (-\partial_t)^m z^{-n-1}$ where $A \in \text{End}_n(\mathbb{C})$ [10, Examples 2.10b, 2.10c].

3.4. Differential algebras

Example 3.8. Recall that the bialgebra X^{cop} is isomorphic to X as an algebra and has the comultiplication $\Delta^{\text{op}}: x \mapsto x_{(2)} \otimes x_{(1)}$. Consider an X^{cop} -differential algebra A , i.e., a topological associative algebra with a left X^{cop} -action such that for $x \in X^{\text{cop}}$, $a, b \in A$

$$x(ab) = (x_{(2)}a)(x_{(1)}b). \quad (3.4)$$

Recall that $\Delta(x) \in X \hat{\otimes} X$ is not, in general, a finite sum. Thus, in order for (3.4) to make sense, we must require that for any $a \in A$, $\text{codim Ann } a < \infty$.

Remark 3.9. For brevity, the above property will never be stated in the further exposition but will always be assumed when we discuss X^{cop} -differential algebras. We will simplify terminology even further and simply call such algebras X^{cop} -algebras, always implying the structure from (3.4).

Remark 3.10. For X such that $\dim F_n X < \infty$ for all n , a typical example of an X^{cop} -algebra is H^{op} . This statement is “dual” to Remark 2.7 and can be deduced in the same way.

We introduce the pseudoalgebra structure on $\text{Diff } A = H \otimes A$. Notice that by H -sesquilinearity it is enough to define the products between elements of the type $1 \otimes a$:

$$(1 \otimes a)_x (1 \otimes b) = 1 \otimes (ax(b)), \quad \text{for any } x \in X. \quad (3.5)$$

Associativity of these products follows from (2.16) and (3.4). Finite codimension of the annihilator of every $a \in A$ implies locality.

Notice that $\text{Diff } A$ is generated over H by $1 \otimes A$. For brevity we will denote elements $1 \otimes a$ by \tilde{a} .

For a free finite H -module V , $\text{Cend } V$ is a differential pseudoalgebra $\text{Diff } H^{\text{op}} \otimes \text{End}_N(\mathbb{C})$. This may be shown directly but in the case of arbitrary H the calculations are cumbersome; this result will follow from a more general statement in the next section (see Theorem 4.9 and Corollary 4.10).

However, the case of $H = \mathbb{C}[\partial]$ is much simpler. Indeed, put $A = \mathbb{C}[\partial_t] \otimes \text{End}_n(\mathbb{C})$ and let the generator t of $X = \mathbb{C}[[t]]$ act as a derivation in ∂_t . For $a \in A$, put $\tilde{a} = \sum at^n z^{-n-1}$. Then $J_A^m = (-1)^m \sum_j (-1)^j \partial^j (\tilde{a}^{(j)})$ where $a^{(j)}$ is the j th derivative of a with respect to ∂_t . Similarly, \tilde{a} can be expressed via J_A^m 's. This shows that $\text{Cend}_n = \text{Diff } A$.

Remark 3.11. A current extension of a differential pseudoalgebra is itself a differential pseudoalgebra. Namely, let H' be a subalgebra of H . Choose a topological basis of $X = \mathbb{C}[[t_1, \dots, t_n]]$ such that $X' = \mathbb{C}[[t_1, \dots, t_r]] = (H')^*$ for some $r < n$. Let A be a differential $(X')^{\text{cop}}$ -algebra. One can consider the induced action of X^{cop} on A , namely let $t_i, i > r$, act on A trivially. Then $\text{Cur}_{H'}^H \text{Diff}_{H'}(A) = \text{Diff}_H A$.

A particular case of the above setting is an arbitrary algebra A with a trivial X^{cop} action. Then $\text{Diff}_H A = \text{Cur}_H A$.

Example 3.12. Here we present a differential pseudoalgebra that is neither current, nor isomorphic to Cend_n .

Recall that the n th Weyl algebra A_n is generated by $\{x_i, y_i\}_{i=1}^n$ such that

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad x_i y_j - y_j x_i = \delta_{ij}.$$

Let $H = \mathbb{C}[\partial_1, \dots, \partial_{2n}]$. Then $X = \mathbb{C}[[t_1, \dots, t_{2n}]]$. Since X is cocommutative, every X -differential algebra gives rise to an H -pseudoalgebra.

To define the action of X on A_n it is enough to describe the action of each t_i and check that it conforms to the Leibniz rule (i.e., that t_i is a derivation of A_n). For $1 \leq i \leq n$ put $t_i = \partial/\partial x_i$ and for $n+1 \leq i \leq 2n$, $t_i = \partial/\partial y_i$. Less formally, we put for $i \leq n$, $t_i = -\text{ad } y_i$, and for $i > n$, $t_i = \text{ad } x_i$; this immediately implies the Leibniz rule.

Remark that as A_n is simple, $\text{Diff}_H A_n$ is also simple (see Lemma 6.1).

This example generalizes to the case of $H = U(\mathfrak{g})$. Let $\hat{\mathfrak{g}}$ be a one-dimensional central extension of \mathfrak{g} : $0 \rightarrow \mathbb{C}c \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$. Let ϕ be the corresponding cocycle. The action of X (or X^{cop}) on \mathfrak{g} extends trivially to $\hat{\mathfrak{g}}$. Put $A = U(\hat{\mathfrak{g}})/(1 - c)$. The X^{cop} -action on $\hat{\mathfrak{g}}$ passes to A . Then $\text{Diff } A$ is a simple H -pseudoalgebra (this can be demonstrated directly or via Lemma 6.1). We will denote it Cend_n^ϕ .

Remark 3.13. When ϕ is the trivial cocycle, $\text{Cend}_n^\phi = \text{Cend}_n$.

For an abelian \mathfrak{g} , $\hat{\mathfrak{g}}$ is the Heisenberg algebra, and A as described above is the Weyl algebra.

4. Unital pseudoalgebras

In this section we will only consider the case of $H = U(\mathfrak{g})$ where \mathfrak{g} is a finite-dimensional Lie algebra. We are interested in some sort of classification of associative H -pseudoalgebras and their representations.

4.1. Definition of unital algebras

Any classification of ordinary algebras begins with that of *unital* algebras. The trivial observation is that an ordinary algebra A is unital (i.e., possesses an identity) if there is an embedding $\mathbb{C} \rightarrow A$ that agrees with the \mathbb{C} -action on A .

We shall introduce a similar concept for H -pseudoalgebras. The role of \mathbb{C} will be played by the “smallest” pseudoalgebra $\text{Cur } \mathbb{C}$ (cf. Lemma 3.5).

In order to define unital pseudoalgebras, we shall study in greater detail the representations of $\text{Cur } \mathbb{C}$. From now on, we will always denote the generator of $\text{Cur } \mathbb{C}$ as an H -module as e .

Recall that $e_1 e = e$ and $e_x e = 0$ for $x \in F_0 X$.

Lemma 4.1.

- (i) Let V be a $\text{Cur } \mathbb{C}$ -module. Then $V = V^0 \oplus V^1$, where V^0 and V^1 are submodules of V such that $e * V^0 = 0$ and for every $v \in V^1$, $e_1 v = v$.
- (ii) V^1 is a torsion-free H -module.

Proof. (i) $\text{Cur } \mathbb{C}$ is an ordinary associative algebra with respect to the product $_1$, hence V splits into the direct sum of ordinary submodules $V^0 \oplus V^1$ such that e acts as a multiplication by i on V^i .

For any $x \in X$, if $e_1 v = 0$, we have $0 = e_x(e_1 v) = (e_{x(2)} e)_{x(1)} v = (e_1 e)_x v = e_x v$, thus $e * V^0 = 0$.

A direct calculation shows that V^0 and V^1 are H -stable.

(ii) Assume that $v \in V^1$ is torsion, i.e., there exists $h \in H$ such that $h v = 0$. Suppose we can choose x such that $S(h)x$ is maximal with respect to e and v . But since

$$0 = e_x(hv) = h_{(2)}(e_{h_{(-1)}x} v) = e_{S(h)x} v \neq 0,$$

this is impossible and for all $x \in X$, $e_x v = 0$. Hence, $v \in V^0 \cap V^1$ and $v = 0$. \square

Remark 4.2. In fact, we will show in the proof of Lemma 5.2 that V^1 is free as an H -module, but for now torsion-freeness will suffice.

Definition 4.3. An H -pseudoalgebra R is called *unital* if

- (1) there exists an embedding $\text{Cur } \mathbb{C} \rightarrow R$;
- (2) as a $\text{Cur } \mathbb{C}$ -module R has no zero component R^0 .

We shall denote the image of the generator of $\text{Cur } \mathbb{C}$ in R by e as well and call it the *pseudoidentity* of R .

Differential pseudoalgebras (see Example 3.8) over unital algebras are unital and, since identity can be adjoined to any ordinary algebra, any differential pseudoalgebra can be embedded into a unital one. Thus, speaking of differential pseudoalgebras, we will always assume them to be unital.

Remark 4.4. It is unknown, in general, what pseudoalgebras can be embedded into unital ones. Torsion-freeness over H is a necessary condition (Lemma 4.6), and one can provide a number of sufficient conditions as well, e.g., having a faithful finite representation (then there is an embedding into Cend_n).

Remark 4.5. Unlike the case of ordinary algebras with an ordinary identity, pseudoidentity is not unique. Consider, for instance, the conformal algebra (i.e., a pseudoalgebra over $\mathbb{C}[\partial]$) $\text{Cur End}_n(\mathbb{C})$. Clearly, $\tilde{1}$ is a pseudoidentity, but so is $\tilde{1} + \partial \tilde{r}$ for any nilpotent r of nilpotency degree 2 (i.e., $r^2 = 0$).

Nonetheless, unital algebras possess a number of good properties.

Lemma 4.6. *Let R be a unital H -pseudoalgebra. Then R is a torsion-free H -module.*

Proof. Lemma 4.1(ii). \square

4.2. Classification

Remark that e_1 acts in $\mathcal{A}(R)$ as a left identity. It is easy to construct an example of a pseudoalgebra such that $\mathcal{A}(R)$ possesses no right identities. E.g., consider $R = \text{Cur } A$ where A has no right identities, then $\mathcal{A}(R)$ is a tensor product of algebras $X \otimes A$. In order to provide a good classification of unital algebras, we need to exclude such degenerate examples.

Definition 4.7. The *left annihilator* $L(R)$ of a pseudoalgebra R is the set of elements a such that $a * b = 0$ for all $b \in R$.

It is clear that $L(R)$ is an ideal of R .

Lemma 4.8. *For a unital R , $L(R) = \{a \mid a * e = 0\}$.*

Proof. Let a be such that $a * e = 0$. For any $x \in X, b \in R$,

$$a_x b = a_x(e_1 b) = (a_{x(2)} e)_{x(1)} b = 0.$$

Hence, $a * b = 0$. \square

Theorem 4.9. *A unital pseudoalgebra R with a zero left annihilator is differential: $R = \text{Diff } A$ for some associative A . Moreover, if R is finitely generated as a pseudoalgebra, A is a finitely generated algebra.*

Proof. Consider the subset $A = 1 \otimes_H R$ of the annihilation algebra $\mathcal{A}(R)$. Clearly, it is a subalgebra of $\mathcal{A}(R)$ with a left identity $1 \otimes e$. We will show that for a unital pseudoalgebra, $R = \text{Diff } A$.

We shall describe, at first, the annihilation subalgebra of $\text{Cur } \mathbb{C}$. Recall that $e * e = (1 \otimes 1) \otimes_H e$. Since $(x \otimes e)(y \otimes e) = xy \otimes e$, e_1 is the identity in $\mathcal{A}(\text{Cur } \mathbb{C})$ and $\mathcal{A}(\text{Cur } \mathbb{C})$ is generated by $\{e_{t_i}\}$.

Suppose R is generated over H by the set R_0 of elements a such that

$$a * e = (1 \otimes 1) \otimes_H a \quad (4.1)$$

(i.e., $a_1 e = a$ and $1 \in X$ is maximal with respect to a and e). Notice that if a is such a generator, then so is $e_x a$ for every $x \in X$.

Clearly $a_x = (a_1 e)_x = a_1 e_x$. As the collection $\{a_x\}$ is unique for each element of R (see Lemma 2.12), we conclude that the above set of generators of R is in 1-1 correspondence with A . Moreover, $a_1 b_1 = (a_1 b)_1$, hence A is an algebra with multiplication determined by the $_1$ -product in R .

Define the action of X on A : $x(a_1) = (e_x a)_1$. The following calculation implies that $x(a_1 b_1) = x_{(2)}(a_1) x_{(1)}(b_1)$:

$$\begin{aligned} e_x(a_1 b) &= (e_{x_{(2)}} a)_{x_{(1)}} b = (e_{x_{(2)}}(a_1 e))_{x_{(1)}} b \\ &= ((e_{x_{(2)}} a)_1 e)_{x_{(1)}} b = (e_{x_{(2)}} a)_1 (e_{x_{(1)}} b). \end{aligned}$$

Therefore, A is an X^{cop} -differential algebra. Hence, as R is torsion free over H , it follows from Lemma 2.12 that the x -products of elements from R_0 satisfy (3.5).

To show that with the X^{cop} action defined as above, $R \simeq \text{Diff } A$, it remains to prove that $R = H \otimes R_0 \simeq H \otimes A$ as H -modules, i.e., that R is a free H -module generated by R_0 . Assume the contrary, namely, that there exist nonzero elements $b_I \in R_0$ such that $\sum_I \partial^I b_I = 0$ for some finite collection of I 's. Among these I 's, choose a maximal J (with respect to the natural ordering of n -tuples). Then $0 = (\sum_I \partial^I b_I)_{t^J} e = (-1)^J b_J$, a contradiction. Hence, $R = H \otimes R_0$, where R_0 is a generating set of R satisfying (4.1). By construction of A , $R = \text{Diff } A$.

Now it remains to construct such a generating set R_0 .

Fix an arbitrary element $a \in R$. For I such that t^I is maximal with respect to a and e , put $a_I = (-1)^I a_{t^I} e$. A direct calculation utilizing (2.8) shows that for any J , $(a_I)_{t^J} e = \delta_{0,J} a_I$, i.e., a_I satisfies (4.1). Consider now the element $a - \partial^I a_I$. For J such that $J \not\leq I$ and $a_{t^J} e = 0$, $\partial^I t^J = 0$, thus clearly, $(a - \partial^I a_I)_{t^J} e = 0$. Also, for $J > I$, $(a - \partial^I a_I)_{t^J} e = -(a_I)_{t^{J-I}} e = 0$. Finally, $(a - \partial^I a_I)_{t^I} e = a_I - (a_I)_1 e = 0$ as well.

We conclude that by subtracting from a elements of the type hb where $h \in H$ and b satisfies (4.1), we can lower the number of I 's such that $a_{t^I} e \neq 0$. Since in the process we also lower the degree of such t^I 's, we will at some point obtain an element c such that $c_{t^I} e = 0$ for $I > 0$. Then either $c_1 e = 0$ as well or $c_1 e \neq 0$. In the former case, $c * e = 0$, hence $c \in L(R)$ and $c = 0$. In the second case, for arbitrary d and x , $c_x d = c_x(e_1 d) = (c_1 e)_x d$, hence $c - c_1 e \in L(R)$ and we see that c satisfies (4.1) as well.

Therefore, given a set of H -generators of R , we can construct a set of generators satisfying (4.1) which shows that R is a differential pseudoalgebra.

Moreover, given a set that generates R as a pseudoalgebra, the elements obtained from it by the above procedure will also generate R and their tensor products with 1 will

generate A . Due to locality, if R is finally generated, we will produce a finite number of generators for A . \square

The above theorem is especially useful in the cases described below:

Corollary 4.10. Cend_n is a differential algebra over $\text{End}_n(\mathbb{C}) \otimes H^{\text{op}}$.

Proof. Definition (3.3) of the pseudoproduct in Cend_n implies that it is unital with a zero left annihilator (the latter also follows from the simplicity of Cend_n , see Corollary 4.11 immediately below), hence it is differential. Repeating the construction of A in the proof of Theorem 4.9 in this particular case gives the description of the underlying X^{cop} -algebra. Namely, we have

$$A = \{1 \otimes_H (1 \otimes b \otimes B) \mid b \in H, B \in \text{End}_n(\mathbb{C})\}$$

with the multiplication

$$(1 \otimes_H (1 \otimes b \otimes B))(1 \otimes_H (1 \otimes c \otimes C)) = 1 \otimes_H (1 \otimes cb \otimes BC),$$

which shows that A is isomorphic to $\text{End}_n(\mathbb{C}) \otimes H^{\text{op}}$. \square

Corollary 4.11. A unital semisimple pseudoalgebra is differential.

Proof. By associativity, the left annihilator is a nilpotent ideal. \square

4.3. Unital pseudoalgebras over cocommutative Hopf algebras

We briefly discuss here what happens in the case of a more general cocommutative H .

Let R be a semisimple unital pseudoalgebra over a cocommutative Hopf algebra H . Recall (Corollary 2.11) that here R is a pseudoalgebra over $H' = U(P(H))$. Interpretation (2.12) of pseudoproduct over H in terms of that over H' together with (2.13) imply that R is unital over H' . In particular, if e is a pseudoidentity over H , it remains such over H' (the calculation is direct and is, therefore, omitted).

Consider now the pseudoproduct $(ge) * e = (g \otimes 1) \otimes_H e$ over H . It does not survive passing to H' (cf. the construction in [3, Chapter 5]), hence $L(R) \neq 0$ by Lemma 4.8. Remark that according to Definition 4.7, it is impossible to find another pseudoidentity in R regarded as an H' -pseudoalgebra such that R would satisfy the conditions of Theorem 4.9. Hence, R is not an H' -differential pseudoalgebra over a unital algebra.

On a more general note, unitality as defined above does not seem to be the right concept for the study of pseudoalgebras over a generic cocommutative algebra: for instance, there is no analogue of Lemma 3.5.

5. Representations of unital pseudoalgebras

We turn now to the description of representations of unital pseudoalgebras. Although we will work only with unital differential pseudoalgebras, in light of Theorem 4.9, this simply means that we impose a technical condition $L(R) = 0$. Since most of the interesting pseudoalgebras are semisimple, this holds automatically (Corollary 4.11).

The goal is to provide a statement similar to Theorem 4.9, i.e., to establish a correspondence between the categories of modules of A and $\text{Diff } A$. As in the previous section, $H = U(\mathfrak{g})$ for a finite-dimensional \mathfrak{g} .

5.1. Structure of representations of unital algebras

Consider a representation V of a differential pseudoalgebra $R = \text{Diff } A$. Recall that by Lemma 4.1, as a $\text{Cur } \mathbb{C}$ -module, $V = V^0 \oplus V^1$, where $e * V^0 = 0$. Thus for any $\tilde{a}, a \in A$ and $v \in V^0$, $\tilde{a}_x v = (\tilde{a}_1 e)_x v = 0$, and $R * V^0 = 0$. Also, since $e_1(a_x v) = (e_1 a)_x v = a_x v$, V^1 is R -stable. Therefore, the decomposition of V is valid over R as well.

Definition 5.1. A module V of a unital differential pseudoalgebra R is *unitary* if it has no zero component V^0 .

Let now $R = \text{Diff } A$ be a unital differential pseudoalgebra and V its unitary module.

As in the proof of Theorem 4.9, for any $v \in V$, we can consider elements $v_I = e_{t^I} v$. If t^I is maximal with respect to e and v , $e_{t^J} v_I = \delta_{0,J} v_I$ for any n -tuple J . Now, consider the difference $w = v - \partial^I v_I$. Direct calculations show that $e_{t^J} w = 0$ for J such that either $J \geq I$ or J is incompatible with I and $e_{t^J} v = 0$. By taking such differences repeatedly we will arrive at w such that $e_{t^I} w = \delta_{0,I} w$. Hence, V is generated over H by elements v such that $e * v = (1 \otimes 1) \otimes_H v$. Denote the set of such as V_0 .

For such an element v , $\tilde{a}_{t^I} v = (\tilde{a}_1 e)_{t^I} v = \tilde{a}_1 (e_{t^I} v)$, hence

$$\tilde{a} * v = (1 \otimes 1) \otimes_H (\tilde{a}_1 v). \quad (5.1)$$

Lemma 5.2. A unitary module of a unital differential pseudoalgebra is free as an H -module.

Proof. We have to show that $V = H \otimes V_0$. Assume the contrary, i.e., the existence of a finite collection of nonzero $v_I \in V_0$ such that $\sum_I \partial^I v_I = 0$. Pick J to be a maximal n -tuple such that $v_J \neq 0$. Then, by calculating $e_{t^J}(\sum_I \partial^I v_I)$ via (5.1), we see that $v_J = 0$. \square

Corollary 5.3. A unital differential pseudoalgebra is free as an H -module.

Now, since V is free over H , we see that $V = H \otimes V_0$, and \tilde{a} acts on V_0 in accordance with (5.1).

Remark that $\mathcal{A}(V) = X \otimes_H V_0 = X \otimes V_0$. In particular, if we write out $\tilde{a}_1 v = v_0 + \sum_i h_i v_i$, where $v_i \in V_0$, then $1 \otimes_H (\tilde{a}_1 v) = 1 \otimes v_0$. Recall that $A = 1 \otimes_H R$ and $a = 1 \otimes_H \tilde{a}$. We can introduce the action of A on V_0 viewed as the subspace $1 \otimes V_0$ of $\mathcal{A}(V)$. Thus,

$$av = a(1 \otimes_H v) = 1 \otimes_H (\tilde{a}_1 v), \quad v \in V_0. \quad (5.2)$$

We sum up the above discussion in the following lemma:

Lemma 5.4. *Let V be a module of a unital differential pseudoalgebra $R = \text{Diff } A$. Then $V = V^0 \oplus V^1$ where $R * V^0 = 0$ and $V^1 = H \otimes V_0$ with the action of R on elements of V_0 described by (5.1). Moreover, there is a structure of an A -module on V_0 described by (5.2).*

5.2. Constructing representations of unital pseudoalgebras

Conversely, let M be a unitary left module for an X^{cop} -differential algebra A . Our goal is to construct a related representation of $R = \text{Diff } A$ on a left H -module $\tilde{M} = H \otimes M$. Before this, we shall endow $X \otimes M$ with the structure of an $\mathcal{A}(R)$ -module. (Even though $\mathcal{A}(R) = X \otimes A$, we will write its basis elements as $x \otimes_H \tilde{a}$ to emphasize the relation with R .)

Naturally, we put $(1 \otimes_H \tilde{a})(1 \otimes m) = (1 \otimes am)$ and $(x \otimes_H e)(1 \otimes m) = (x \otimes m)$. In general, $(x \otimes_H \tilde{a})(y \otimes m) = (1 \otimes_H \tilde{a})(xy \otimes_H e)(1 \otimes m) = (xy \otimes_H \tilde{a})(1 \otimes m)$. Hence, to describe explicitly the action of $\mathcal{A}(R)$ on $X \otimes M$, it suffices to write out the expression for $(x \otimes_H \tilde{a})(1 \otimes m)$. Recall that $x \otimes_H \tilde{a}$ is $\tilde{a}_x = \tilde{a}_1 e_x = (e_1 \tilde{a})_x$. Using (2.16) and (2.17), it is not difficult to see that

$$\tilde{a}_1 e_x = \sum_I (e_{(x_{(2)} \partial^I)_t} \tilde{a})_{x_{(1)}} = \sum_I (\partial^I e)_x (e_t \tilde{a})_1,$$

where the first equality is valid because $\sum_I (t^J \partial^I) t^I = 0$ whenever $J \neq 0$ (cf. [3, Lemma 2.2]).

Therefore, $x \otimes_H \tilde{a} = \sum_I (x \partial^I \otimes_H e)(1 \otimes_H t^I \tilde{a})$, and we obtain

$$(x \otimes_H \tilde{a})(1 \otimes m) = \sum_I (\partial^I x) \otimes (t^I (a)m). \quad (5.3)$$

We now turn to the description of \tilde{M} . First of all, remark that $\mathcal{A}(\tilde{M}) = X \otimes M$. Thus, for $x, y \in X$, $(e_x \tilde{m})_y = e_{x_{(2)}} \tilde{m}_{x_{(1)}y} = (x_{(2)} x_{(-1)} y) \otimes m = \varepsilon(x) y \otimes m$ (by (2.3)). Hence, if $\varepsilon(x) = 0$, $e_x m = 0$, and $e * \tilde{m} = (1 \otimes 1) \otimes_H \tilde{m}$. That is, \tilde{M} is a unitary module and, according to (5.1), $\tilde{a} * \tilde{m} = (1 \otimes 1) \otimes_H \tilde{a}_1 \tilde{m}$. It remains only to determine $\tilde{a}_1 \tilde{m}$ for arbitrary $a \in A, m \in M$. Checking the coefficients, we obtain from (5.3):

$$\tilde{a}_1 \tilde{m} = \sum_I \partial^I (t^I (\widetilde{am})). \quad (5.4)$$

We summarize the above discussion as

Lemma 5.5. *Let A be a unital X^{cop} differential algebra, and M an A -module. Then $\tilde{M} = H \otimes M$ is a representation of $\text{Diff } A$ with the action described by*

$$\tilde{a} * \tilde{m} = (1 \otimes 1) \otimes_H \left(\sum_I \partial^I (\widetilde{t^I(a)m}) \right). \quad (5.5)$$

5.3. Classification and corollaries

Summing up, we obtain the full description of representations of unital differential pseudoalgebras.

Theorem 5.6. *Let V be a module of a unital differential pseudoalgebra $R = \text{Diff } A$. Then $V = V^0 \oplus V^1$, where $R * V^0 = 0$ and $V^1 = \tilde{M}$ for some A -module M . In particular, V^1 is free over H .*

Proof. The decomposition of V as well as the 0 action of R on V^0 follow from Lemma 5.4. Freeness of V^1 is explained in Lemma 5.2, in particular, we know (again, from Lemma 5.4) that $V^1 = H \otimes V_0$, where V_0 is an A -module.

We can construct another representation \tilde{V}_0 of R . By comparing (5.1) with (5.5), we see that the R -action on both V^1 and \tilde{V}_0 is determined by the 1 -action only. Now, define the degree of $a \in A$ as the maximal value of $|I|$ such that $t^I(a) \neq 0$. Inducting on the degree and comparing (5.2) with (5.4), we conclude that $\tilde{V}_0 \simeq V^1$. \square

Remark 5.7. The proofs of both Theorems 4.9 and 5.6 required constructions of particular H -generating sets of, respectively, a given pseudoalgebra and a given module. However, if one considers a unital algebra as a $\text{Cur } \mathbb{C}$ -module, these bases are clearly different (e.g., compare (4.1) and (5.1)). For conformal algebras Cend_n both were written out explicitly in Example 3.8.

For the general case of pseudoalgebras Cend_n , the basis from Example 3.7 is the one corresponding to its structure as a $\text{Cur } \mathbb{C}$ -module: $\text{Cend}_n = \tilde{M}_n$ where $M_n = H \otimes \text{End}_n(\mathbb{C})$.

Remark 5.8. Clearly, a non-unitary A -module M gives rise to a non-unitary $\text{Diff } A$ -module $\tilde{M} = H \otimes M$. However, in this case the converse is not true. For example, a non-unitary $\text{Diff } A$ -module does not have to be free over H .

Nonetheless, for consistency we will sometimes use the notation \tilde{M} for non-unitary modules. In particular, we will denote the zero-dimensional $\text{Diff } A$ -module as $\tilde{0}$.

We now turn to the structural theory of representations of unital algebras; obviously, Theorem 5.6 will be our main tool.

The definitions are the same as in the ordinary case. A module V over a pseudoalgebra is called *irreducible* if it contains no submodules except for 0 and V , *indecomposable* if it can not be presented as a sum of two nonzero submodules, and *completely reducible* if it decomposes into a direct sum of irreducible modules.

Corollary 5.9. *Let \tilde{M} be a unitary module of a unital differential pseudoalgebra $R = \text{Diff } A$ and W its submodule. Then $W = \tilde{N}$ for an A -module $N \subset M$.*

Proof. The argument is the same as in the proof of Lemma 5.2.

Put $N = \{m \mid \tilde{m} \in W\}$. Obviously, $\tilde{N} \subset W$. Theorem 5.6 implies that W is unital as well, so we may apply (5.1). For an arbitrary element $w = \sum_I \partial^I \tilde{m}_I \in W$, $m_I \in M$, let J be a maximal n -tuple among I 's such that $m_I \neq 0$. Then $e_{I,J} w = (-1)^J m_J \in W$. By induction, we obtain that all m_I lie in N , and $w \in \tilde{N}$. \square

Corollary 5.10. *Let V be a module over a unital differential pseudoalgebra $R = \text{Diff } A$. Then V is irreducible if and only if $V = \tilde{M}$ for an irreducible A -module M or $V = V^0$.*

Proof. Since $V = V^0 \oplus V^1$, either of the components must be 0. If $V = V^1$, then by Theorem 5.6, $V = \tilde{M}$ and, clearly, M must be irreducible as well. If $V = V^0$, then every element of V gives rise to an R -submodule, i.e., $V = \tilde{0}$.

Conversely, by Corollary 5.9, a nonzero irreducible A -module M gives rise to an irreducible R -module \tilde{M} . \square

Similarly, we can prove:

Corollary 5.11. *Let V be a module over a unital differential pseudoalgebra R . Then V is indecomposable if and only if $V = \tilde{M}$ for an indecomposable A -module M (not necessarily nonzero).*

Corollary 5.12. *Let V be a module over a unital differential pseudoalgebra R . Then V is completely reducible if and only if $V = \tilde{M}$ for a completely reducible A -module M .*

Proof. Clearly, V^0 is completely reducible if and only if $V^0 = \tilde{0}$. Complete reducibility of V^1 again follows from Theorem 5.6 and Corollary 5.9. \square

Remark 5.13. The only major notion that does not immediately carry over from A -modules to $\text{Diff } A$ -modules is faithfulness. Indeed, if \tilde{M} is faithful, M need not be. The right concept here is to require that the annihilator of M does not contain any X^{cop} -stable ideals.

5.4. Representations of pseudolinear conformal algebras

The above corollaries allow us, for example, to classify the representations of the pseudoalgebra Cend_n . Below we shall do so in the case of conformal algebras (see Example 3.6).

We will completely describe irreducible and indecomposable modules over the conformal algebra $\mathfrak{W}_n = \text{Cend}_n$. By the above corollaries this comes down to explaining how $\text{End}_n(\mathbb{C}) \otimes \mathbb{C}[\partial]$ -modules look like.

In Example 3.7 we described the standard module E_n over \mathfrak{W}_n . By Corollary 4.10 $\text{Cend}_n = \text{Diff } A$ where A is the algebra of $n \times n$ matrices over $\mathbb{C}[\partial]$. Thus $E_n = \tilde{M}_n$ where the A -module M_n is an n -dimensional vector space on which ∂ acts as the identity operator.

This can be generalized to the case of the module M_n^α which is again an n -dimensional space on which ∂ acts as $\alpha \in \mathbb{C}$. Thus we obtain a family of modules $E_n^\alpha = \widetilde{M}_n^\alpha$ which can be explicitly written as

$$E_n^\alpha = \left\{ a(z) = \sum (at^n e^{-\alpha t}) z^{-n-1}, a \in \mathbb{C}^n \right\}, \quad \alpha \in \mathbb{C}.$$

Every irreducible $\mathbb{C}[\partial]$ -module is one-dimensional (M_1^α in the above notations) and every irreducible $\text{End}_n(\mathbb{C}) \otimes \mathbb{C}[\partial]$ -module is of the form M_n^α . Thus, Corollary 5.10 implies

Proposition 5.14. *Finite irreducible modules over \mathfrak{W}_n are of the form E_n^α with the standard action.*

The case of indecomposable modules is similar. Let U be a finite-dimensional space with an indecomposable endomorphism α . Then U is an indecomposable $\mathbb{C}[\partial]$ -module ($\partial \mapsto \alpha$, $\alpha \in \text{End}(U)$) and every indecomposable $\mathbb{C}[\partial]$ -module has this form. For $\text{End}_n(\mathbb{C}) \otimes \mathbb{C}[\partial]$ the module $M_n^\alpha(U) = \mathbb{C}^n \otimes U$ with the obviously defined action is indecomposable. Remark that the irreducible modules M_n^α defined above are simply $M_n^\alpha(\mathbb{C})$.

These describe all unital indecomposable modules over $\text{End}_n(\mathbb{C}) \otimes \mathbb{C}[\partial]$ (consider the decomposition of such as a $\mathbb{C}[\partial]$ -module; all components will be of the same height). Observe also that for a finite indecomposable module V , V^0 is necessarily 0. We thus obtain indecomposable \mathfrak{W}_n modules

$$E_n^\alpha(U) = \widetilde{M}_n^\alpha(U).$$

In [11] they were denoted $\sigma_\alpha^{\text{as}}$ (“as” stands for associative); the following result was first stated there as well:

Proposition 5.15. *Finite indecomposable modules over \mathfrak{W}_n are exactly $E_n^\alpha(U)$.*

Remark 5.16. After this paper was completed, another approach to classification of modules of Cend_n appeared in [1]. Roughly speaking, first the irreducible representations are classified and then one has to study their extensions (as $\mathbb{C}[\partial]$ -modules). Although unitality is never required (indeed, [1] also describes modules over an important class of subalgebras of Cend_n), the proof is quite technical.

In the case of Cend_n , module extensions can be easily described within the framework of this paper. Indeed, let

$$0 \rightarrow \tilde{U} \rightarrow \tilde{V} \rightarrow \tilde{W} \rightarrow 0$$

be a short exact sequence of Cend_n -modules (the maps are taken over $\mathbb{C}[\partial]$). As a sequence of $\mathbb{C}[\partial]$ -modules, it splits. Therefore, we have $\tilde{V} = \tilde{U} \oplus (\tilde{U})^\perp$, where $(\tilde{U})^\perp \simeq \tilde{W}$ over $\mathbb{C}[\partial]$. For any $v \in V$, we have $\tilde{v} = \tilde{u} + \sum_0^n \partial^j \tilde{w}_j$. Apply $e_{(n)}$ to \tilde{v} and consider the

coefficient at ∂^n . If $n > 0$, we obtain $\widetilde{w}_n = 0$, thus $\widetilde{v} = \widetilde{u} + \widetilde{w}$ and we obtain a short exact sequence

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0.$$

This agrees with the general philosophy that every property of modules over a differential pseudoalgebra R comes from the corresponding property of modules over R . The calculation in [1] for modules over non-unital algebras demonstrates aptly that this is not true in the general case.

6. Simple unital pseudoalgebras

In this section we classify finitely generated simple unital pseudoalgebras satisfying certain conditions. The motivation is to find pseudoalgebras similar to the most important simple pseudoalgebras, Cend_n .

6.1. General results

As we are interested in simple unital pseudoalgebras, our objects of study are necessarily differential (Corollary 4.11). For such a pseudoalgebra $R = \text{Diff } A$, simplicity easily translates into a property of the X^{cop} -algebra A . Namely, call A X^{cop} -simple if it contains no nonzero ideals stable under the action of X^{cop} . Then we have

Lemma 6.1. *Diff A is simple if and only if A is X^{cop} -simple.*

Proof. Let I be an X^{cop} -stable ideal of A . Put \widetilde{I} to be the H -submodule of $\text{Diff } A$ generated by $\{\widetilde{a} \mid a \in I\}$. Then by (3.5) and Corollary 5.3, \widetilde{I} is an ideal of $\text{Diff } A$.

Conversely, let J be a nonzero ideal of $\text{Diff } A$. For $b = \sum_K \partial^K \widetilde{a}_K \in J$, pick L maximal among indices such that $a_L \neq 0$ and consider the product $b_{\partial^L} e$. It equals $(-1)^L \widetilde{a}_L$ (see the proof of Theorem 4.9). By induction we obtain that all \widetilde{a}_K belong to J (compare this to the proof of Corollary 5.9). As $\widetilde{a}_1 \widetilde{b} = \widetilde{a} b$, this implies that $\widetilde{J} = \{a \mid \widetilde{a} \in J\}$ is a nonzero ideal of A . As

$$e_x \widetilde{a} = x(\widetilde{a}) \in J,$$

this ideal is X^{cop} -stable. \square

In fact, we proved a more general result:

Lemma 6.2. *The lattice of X^{cop} -stable ideals of A is isomorphic to the lattice of ideals of $\text{Diff } A$.*

Proof. Indeed, in notations from the proof of Lemma 6.1, it is clear that $\widetilde{I} = I$ and $\widetilde{J} = J$ for any X^{cop} -stable ideal I of A and any ideal J of $\text{Diff } A$. \square

6.2. Small X^{cop} -algebras

Let A be an X^{cop} -algebra. We introduce a filtration on A :

$$F^n A = (F_n X)^\perp = \{a \in A \mid F_n X(a) = 0\}. \quad (6.1)$$

Because of (2.9) and (3.4), the filtration (6.1) respects multiplication in A :

$$(F^m A)(F^n A) \subset F^{m+n} A. \quad (6.2)$$

Remark 6.3. Since the annihilator of every element of A is not empty, $A = \bigcup_n F^n A$.

We say that a nonzero $a \in A$ has *degree* m if $a \in F^m A \setminus F^{m-1} A$.

Remark that the action of t_i lowers the degree, i.e., $t_i(F^{m+1} A) \subset F^m A$, otherwise $t_i^{m+1} F^{m+1} A \neq 0$. Notice also the following useful properties:

Lemma 6.4.

- (i) If $\deg a = m$, then for any M such that $|M| = m$, $t^M(a) \in F^0 A$.
- (ii) If $\deg a = m$, there exists M with $|M| = m$ such that $t^M(a) \neq 0$.
- (iii) $\deg(ab) \leq \deg(a) + \deg(b)$.

Proof. (i), (ii) follow immediately from (6.1); whereas (iii) is a reformulation of (6.2). \square

By definition, for H (or, rather, for H^{op}) this filtration coincides with the canonical one. Here every filtration component is finite-dimensional.

Lemma 6.5. Let A be an X^{cop} -algebra such that $\dim F^0 A < \infty$. Then every filtration component is finite-dimensional.

Proof. For any $a \in F^{m+1} A \setminus F^m A$, there exists t_i such that $t_i(a) \in F^m A \setminus F^{m-1} A$. Hence,

$$F^{m+1} A = \bigcup_{i=1}^n t_i^{-1}(F^m A). \quad (6.3)$$

Notice also that filtration (6.1) can be introduced for any X -module M as long as all its elements have nonzero annihilators. In particular, this is true for submodules M_i of A defined by $M_i = \{a \in A \mid t_i a = 0\}$.

Consider the particular case of $n = 1$ (i.e., $X = \mathbb{C}[[t]]$). To demonstrate the statement of the lemma, we induct on m . For any two elements $a, b \in t^{-1}(F^m A) = F^{m+1} A$ such that $t(a) = t(b)$, we have $a - b \in F^0 A$. Thus $\dim F^{m+1} A \leq (\dim F^m A)(\dim F^0 A) \leq (\dim F^0 A)^{m+1}$ and $F^m A$ is finite-dimensional for all m .

Now we induct on n . Let $a, b \in t_i^{-1}(c)$ for some i and $c \in F^m A$. Then $a - b \in F^m M_i$. Hence (6.3) implies:

$$\dim F^{m+1} A \leq \sum_{i=1}^n \dim t_i^{-1}(F^m A) \leq \sum_{i=1}^n (\dim F^m A)(\dim F^m M_i).$$

Since by induction $\dim F^m M_i < \infty$, we see that $F^{m+1} A$ is finite-dimensional. \square

Remark 6.6. One of the results in [15] can be reinterpreted as follows: when $H = \mathbb{C}[\partial]$, every finitely generated X^{cop} -simple X^{cop} -algebra of GKdim (Gelfand–Kirillov dimension, see [13]) not exceeding 1 has finite filtration components.

However, this is not true in general and, by itself, limiting $\text{GKdim } A$ does not imply any similarity of $\text{Diff } A$ to Cend_n as we will see immediately below. Remark, first, that for a current pseudoalgebra $\text{Cur } A$, the filtration is trivial: $F^0 A = A$. Unlike the case of conformal algebras ($H = \mathbb{C}[\partial]$), for larger H there exist non-finite finitely generated simple current algebras $\text{Cur } A$ such that $\text{GKdim } A \leq \text{GKdim } H$, see, e.g., Example 3.12. Moreover, consider the following example:

Example 6.7. Let $H = U(\mathfrak{g})$ where \mathfrak{g} is the three-dimensional abelian algebra. Then $X = \mathbb{C}[[t_1, t_2, t_3]]$ and $X = X^{\text{cop}}$. Let $A = \mathbb{C}[x, y]$ with the action of X defined by $t_1 = \partial/\partial x$, $t_2 = 0$, $t_3 = 0$. Then $F^0 A = \mathbb{C}[y]$ and we obtain an X^{cop} -algebra that is smaller than H but has a non-trivial filtration with infinite filtration components. One can replace A with the Weyl algebra $A_1 = \mathbb{C}\langle x, y \mid xy - yx = 1 \rangle$ (see Example 3.12) and obtain an X^{cop} -simple algebra smaller than H that has a non-trivial filtration with infinite filtration components.

Thus, pseudoalgebras similar to Cend_n should not be described by a simple combinatorial condition such as a bound on GKdim ; although, some sort of a growth restriction should be imposed.

We arrive at the following definition:

Definition 6.8. An X^{cop} -algebra A is called *small* if $\dim F^n A < \infty$ for all n .

Thus, Lemma 6.5 can be reformulated as

Lemma 6.9. Let A be an X^{cop} -algebra. If $F^0 A$ is finite-dimensional, then A is small.

Remark 6.10. It will follow from Theorem 6.23 that for a simple finitely generated pseudoalgebra $\text{Diff } A$ with a small A , A has Gelfand–Kirillov dimension not exceeding $\text{GKdim } H = \dim \mathfrak{g}$. This is a generalization of the converse of the first statement in Remark 6.6.

6.3. Digression: simplicity conditions for small X^{cop} -algebras

In the next subsection we will show at first that under certain conditions A can be encoded by an associative algebra $(F^0 A)$ and a certain Lie algebra acting on it. These conditions will be automatically satisfied when A is small and X^{cop} -simple and $F^0 A$ is simple. These two statements about simplicity of A or $F^0 A$ are closely related.

Indeed, let A be a small X^{cop} -algebra. Clearly, if J is a nonzero proper X^{cop} -stable ideal of A , $J \cap F^0 A$ is a nonzero ideal of $F^0 A$ by Lemma 6.4(ii).

Example 6.11. Let $A = \text{End}_2(\mathbb{C})$ with an inner derivation

$$\delta = \text{ad} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$F^0 A = \mathbb{C} + \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore, A may be simple while $F^0 A$ is not.

This does not happen when $F^0 A$ semisimple. In this case, there exist nonzero idempotents $a, b \in F^0 A$ such that $a(F^0 A)b = 0$. If $aAb = 0$, then AaA is a proper X^{cop} -stable ideal of A , and A is not X^{cop} -simple. However, if there exists $c \in A$ such that $acb \neq 0$, then by applying a suitable element of X , we may assume that $acb \in F^0 A$. But as $acb = a(acb)b = 0$, we obtain a contradiction.

The above example suggests that the case of an inner derivation is different from the general one. This is also true on the pseudoalgebra level, where the action of an inner derivation can be changed to a trivial one by changing the pseudoidentity [15]. This leads us to conjecture that whenever the action of X^{cop} is external in some sense, A is X^{cop} -simple if and only if $F^0 A$ is simple. When X is cocommutative, “external” should be understood in the sense of [12]: $X \not\rightarrow \text{Der } A_f$, where A_f is the Martindale quotient of A .

6.4. Simple small X^{cop} -algebras

For the rest of this subsection, A will always stand for a simple small X^{cop} -algebra with $F^0 A$ simple (although simplicity is not always necessary for the statements below to hold).

Assume now that A satisfies the following technical condition:

Condition 6.12. As an $F^0 A$ -module, $F^1 A$ is generated by 1 and elements b_i , $1 \leq i \leq r$, where $r \leq n$, such that $t_j(b_i) = \delta_{ij}$.

Remark 6.13. If $F^1 A$ is generated over $F^0 A$ by elements b_i , $i \in \mathcal{I}$, where $\mathcal{I} \subset \{1, \dots, n\}$ such that $t_j(b_i) = \delta_{ij}$, we can always renumerate t_i 's, so that Condition 6.12 holds.

Moreover, we will show below (Lemma 6.19) that in this case A is an algebra over $\mathbb{C}[[t_1, \dots, t_r]]^{\text{cop}}$, i.e., that $\mathbb{C}[[t_1, \dots, t_r]]$ is closed under the action of Δ .

We will show in Theorem 6.23 that simplicity of A implies the above condition (if, of course, $A \neq F^0 A$). The proof is simple but lengthy, hence we delay it and turn to demonstrating the consequences of Condition 6.12. If it holds, the structure of A is remarkably nice. Namely,

Lemma 6.14. *If Condition 6.12 holds, A is generated by $F^1 A$. In particular, A is finitely generated.*

Lemma 6.15. *If Condition 6.12 holds, $\text{Span}(F^0 A, b_i)$ is a Lie subalgebra \mathfrak{b} of $A^{(-)}$.*

Lemma 6.16. *If Condition 6.12 holds, $[\mathfrak{b}, F^0 A] \subset F^0 A$.*

The proofs of the last two statements are immediate and the proof of the first comes down to solving a system of linear differential equations.

Proof of 6.16. For any $a \in F^0 A$ and any j , $t_j([b_i, a]) = [t_j(b_i), a] = 0$. \square

We can go further and provide a complete description of \mathfrak{b} . By Lemma 6.16, $\text{ad } b_i$ is a derivative of $F^0 A$ which is a finite-dimensional simple algebra. Hence, it is inner, i.e., $\text{ad } b_i = \text{ad } c_i$ [8]. We can replace b_i with $b_i - c_i$, then the span of b_i 's will act trivially on $F^0 A$. It follows that for any b_i, b_j , we have $[b_i, b_j] \in \text{Span}_{\mathbb{C}}(b_i) + Z(F^0 A)$. More explicitly,

Lemma 6.17. *If Condition 6.12 holds, we can choose b_i 's, so that for any $1 \leq i, j \leq r$,*

$$[b_i, b_j] = \sum_{k=1}^r c_{ij}^k b_k + a_{ij}, \quad \text{where } c_{ij}^k, a_{ij} \in \mathbb{C}, \quad (6.4)$$

and $[b_i, F^0 A] = 0$ for all i .

We can finally turn to Lemma 6.15.

Proof of 6.15. For any $a \in F^1 A$ and any j , by (2.8), $t_j([a, b_i]) = [t_j(a), b_i] + [a, t_j(b_i)] \bmod F^0 A$. Hence, by Lemma 6.16, $t_j([a, b_i]) \in F^0 A$ for all j . \square

Proof of 6.14. Let B be the subalgebra of A generated by \mathfrak{b} as defined in the statement of Lemma 6.15.

We will prove the following three statements simultaneously:

- (i) For all m , $F^m A \subset B$;
- (ii) For all m and any $k > r$, $t_k F^m A \subset F^{m-2} A$;
- (iii) For any collection $\{c_i\}_{i=1}^r$ of elements from $F^{m-1} A$ such that $t_i(c_j) = t_j(c_i)$ for all i, j , there exists $c \in F^m A$ such that $t_i(c) = c_i$.

Clearly (i) will imply the statement of the lemma.

Remark first that by Condition 6.12, for $m = 1$ (i) and (ii) hold automatically. As for (iii), for $m = 1$ we let $c = \sum_i c_i b_i$.

We first demonstrate (ii): let $a \in F^m A$. For any j , $t_j t_k(a) = t_k t_j(a) \in t_k F^{m-1} A \subset F^{m-3} A$ and we are done.

Now assume by induction that (iii) holds for $m - 1$. Let $\{c_i\}$ be a collection of elements from $F^m A$. Here and below we will always take $i \leq r$. By additivity of action of X we may assume $\deg c_i = m$ for all i . When X is cocommutative, t_i 's act simply as derivatives, thus (iii) comes down to solving a system of linear differential equations. Moreover, by induction, first we can pass to $\text{gr } A$, i.e., solve the system modulo $F^{m-1} A$. By Lemma 6.17 it is equivalent to assuming that b_i 's commute with each other and elements of $F^0 A$. Then the result is classical.

By (2.8), if we consider the action of t_i 's modulo $F^{m-1} A$, there is no difference between the general and the cocommutative case (i.e., the solution for c obtained above is valid modulo $F^{m-1} A$). Hence, there exists an element $c' \in F^{m+1} A$ such that $t_i(c') = c_i + d_i$ where $d_i \in F^{m-1} A$. Let d be an element from $F^m A$ such that $t_i(d_i) = d$. Then for $c = c' - d$, $t_i(c) = c_i$.

We turn to (i). Assume by induction that $F^m A \subset B$. For an arbitrary a of degree $m + 1$, put $t_i(a) = a_i \in F^m A$. The collection $\{a_i\}$ satisfies the conditions of (iii); therefore, we can produce an element $c \in F^{m+1} A$ such that $t_i(c) = a_i$. By construction, $c \in B$ as $b_i \in B$ and $a_i, d \in F^m A$. Thus, (ii) implies that $t_j(a - c) \in F^{m-1} A$ for all j and $a - c \in F^m A$. Therefore, $a \in B$. \square

Corollary 6.18. *If for $a \in A$ and all $i \leq r$, $t_i(a) = 0$, then $a \in F^0 A$.*

Proof. By Lemma 6.14, a is the sum of monomials of the type $cb_{i_1} \dots b_{i_m}$, $c \in F^0 A$. Denote $\deg a$ by m . Pick a monomial of degree m , say, it ends with b_j .

We may rewrite (2.8) as $\Delta(t_j) = 1 \otimes t_j + t_j \otimes 1 + \sum_k t_k \otimes y_{jk} + \text{summands that have first terms of degree greater than 1}$, where $y_{jk} \in X$ has no constant terms. Then it is clear that monomials of the highest degree in $t_j(a)$ come from monomials in a of degree m . Hence, $\deg t_i(a) = m - 1$.

It follows that if a satisfies the statement of the corollary, $a \in F^1 A$. Condition 6.12 forces $a \in F^0 A$. \square

We conclude that A is generated by a simple associative algebra $F^0 A$ and a Lie subalgebra of $A^{(-)}$ that acts trivially on $F^0 A$. Our goal now is to describe this subalgebra, i.e., to explain the structural constants c_{ij}^k in (6.4).

Let $i < j \leq r$. Clearly, $c_{ij}^k = t_k(b_i b_j - b_j b_i)$ for $k \leq r$. For $k > r$, we put $c_{ij}^k = 0$. Consider now the structural constants of \mathfrak{g} : $[\partial_j, \partial_i] = \sum d_{ji}^k \partial_k$. By definition, $d_{ji}^k = \langle t_k, [\partial_j, \partial_i] \rangle$.

Since $\partial_i \partial_j$ is an element of the PBW-basis of H , $\langle t_k, \partial_i \partial_j \rangle = 0$, and we have $d_{ji}^k = \langle t_k, \partial_j \partial_i \rangle = \langle t_{k(1)}, \partial_j \rangle \langle t_{k(2)}, \partial_i \rangle$. Therefore, the only summand of $\Delta(t_k)$ proportional to $t_j \otimes t_i$ is $d_{ji}^k t_j \otimes t_i$ (and if $d_{ji}^k = 0$, there is no such summand). Remark also that $\Delta(t_k)$ has no summand proportional to $t_i \otimes t_j$, otherwise $\langle t_k, \partial_i \partial_j \rangle \neq 0$.

The X^{cop} -action on A implies that

$$c_{ij}^k = \langle t_{k(2)}, b_i \rangle \langle t_{k(1)}, b_j \rangle - \langle t_{k(2)}, b_j \rangle \langle t_{k(1)}, b_i \rangle = d_{ij}^k.$$

In particular, we have

Lemma 6.19. *If Condition 6.12 holds, $\text{Span}(\partial_1, \dots, \partial_r)$ is a Lie subalgebra of \mathfrak{g} .*

In the same way as above it is not difficult to prove that for $k > r$ in (2.8), whenever both $t^{K_j}, t^{L_j} \in \mathbb{C}[[t_1, \dots, t_r]]$, $c_j = 0$. Thus, by induction, using the formula for $\Delta(t_k)$ stated in the proof of Corollary 6.18, we have

Lemma 6.20. *For $k > r$, t_k acts as 0 on A .*

We can also strengthen the statement of Lemma 6.4(ii):

Corollary 6.21. *If $\deg a = m$, there exists a unique M with $|M| = m$ such that $t^M(a) \neq 0$.*

Now denote subalgebra $\text{Span}(\partial_1, \dots, \partial_r)$ by \mathfrak{h} . We can pass to H^{op} and consider its Lie subalgebra also spanned by $\partial_1, \dots, \partial_r$; denote it by \mathfrak{h}^{op} . Taking into account that a_{ij} 's in (6.4) need not be 0, we obtain

Lemma 6.22. *If Condition 6.12 holds, b_i 's generate a Lie subalgebra of A isomorphic either to \mathfrak{h}^{op} , $\mathfrak{h} \subset \mathfrak{g}$, or its non-trivial 1-dimensional abelian extension.*

Theorem 6.23. *Let A be an X^{cop} -simple small X^{cop} -algebra such that $F^0 A$ is simple. Then A is isomorphic to either of*

- (★) $\text{End}_n(\mathbb{C})$ with a trivial X^{cop} -action;
- (★★) $\text{End}_n(\mathbb{C}) \otimes U(\mathfrak{h}^{\text{op}})$, where \mathfrak{h} is a Lie subalgebra of \mathfrak{g} , and the action of X^{cop} is determined by the action of $U(\mathfrak{h})^*$;
- (★★★) $\text{End}_n(\mathbb{C}) \otimes (U(\widehat{\mathfrak{h}^{\text{op}}})/(1 - c))$, where $\widehat{\mathfrak{h}^{\text{op}}}$ is a 1-dimensional abelian extension $1 \rightarrow \mathbb{C}c \rightarrow \widehat{\mathfrak{h}^{\text{op}}} \rightarrow \mathfrak{h}^{\text{op}} \rightarrow 1$ of \mathfrak{h}^{op} for a Lie subalgebra \mathfrak{h} of \mathfrak{g} , and the action of X^{cop} is determined by the action of $U(\mathfrak{h})^*$.

Moreover, in the last two cases A is a simple $(U(\mathfrak{h})^*)^{\text{cop}}$ -algebra.

Proof. $F^0 A = \text{End}_n(\mathbb{C})$. If $A \neq F^0 A$, assume that Condition 6.12 holds. Then Lemma 6.22 implies that $F^1 A / F^0 A$ is isomorphic to either $(\text{End}_n(\mathbb{C}) \otimes \widehat{\mathfrak{h}})/(1 - c)$ or $\text{End}_n(\mathbb{C}) \otimes \mathfrak{h}$, and by Lemma 6.14, $F^1 A$ generates all of A .

Therefore, there exists a natural surjective map ϕ of either the algebra of type (★★) or (★★★) onto A , which is an isomorphism on the first filtration component. Notice that these algebras are small and satisfy Condition 6.12. Notice also that ϕ commutes with the action of X^{cop} , in particular, it preserves the filtration (6.1). To prove injectivity, let a be the element of least degree such that $\phi(a) = 0$. By statement (iii) of the proof of

Lemma 6.14, there exists $c \in A$ such that $t_i(c) = \phi t_i(a)$, $i \leq r$. Let c' be a preimage of c under ϕ . Then by Corollary 6.18, $c' - a$ lie in the zero component and $\phi(a) \in \phi(c') + F^0 A$, a contradiction.

The last claim of the theorem follows from Lemma 6.20.

It remains to show that Condition 6.12 is valid for simple small X^{cop} -algebras such that $F^1 A \neq \emptyset$ and $F^0 A$ is simple.

Remark that we can change the basis of \mathfrak{g} , hence, the generating set of X . Thus we will abandon the notation t_1, \dots, t_n that stands for a fixed generating set and will work with elements of X of degree 1. Let $T_0 = \{t \in X \mid \deg t = 1, t(F^1 A) = 0\}$.

Clearly, for any $t \notin T_0$, there exists $b \in F^1 A$, $t(b) \neq 0$. Moreover, if $t \notin T_0$, there exists b_t such that $t(b_t) = 1$. Indeed, let $b \in F^1 A$ be such that $t(b) \neq 0$. Then since $F^0 A$ is simple, there exist s_{1k}, s_{2k} such that $t(\sum_k s_{1k} b s_{2k}) = \sum_k s_{1k} t(b) s_{2k} = 1$.

In some sense b_t , as defined above, is unique. Put $N(t) = \{b \mid t(b) = 0\}$. Then for any b , $b - t(b)b_t \in N(t)$ and $\text{rk } F^1 A / N(t) = 1$.

Pick an arbitrary element $t \in X$ of degree 1, $t \notin T_0$. Let b'_1 be such that $t_1(b'_1) \neq 0$, and, inductively, b'_j an element from $\bigcap_{i=1}^{j-1} N(t_i)$ such that there exists t_j for which $t_j(b'_j) \neq 0$. In this way we obtain sequences b'_1, \dots, b'_m and t_1, \dots, t_m . The process terminates when $\bigcap_{i=1}^m N(t_i) = F^0 A$, i.e., when it is annihilated by all t 's. We may assume that $t_i(b'_i) = 1$. Now let $b_m = b'_m$ and, inductively, $b_i = b'_i - \sum_{j>i} t_j(b'_j)b_j$. In this way we obtain b_i 's such that $t_j(b_i) = \delta_{ij}$ for $i, j \leq m$.

For an arbitrary $b \in F^1 A \setminus F^0 A$, consider the difference $b - \sum_{i=1}^m t_i(b)b_i$. As it is killed by all t_i 's, by construction it lies in $F^0 A$, hence $1, b_1, \dots, b_m$ is the basis of $F^1 A$ over $F^0 A$. Notice that this is also true when we consider $F^1 A$ as a right $F^0 A$ -module.

For any $t \notin T_0$ consider the operator $\sum_{i=1}^m t(b_i)t_i$ on $F^1 A$. Using the left and right bases constructed above, it is easy to see that it acts exactly like t .

As in the proof of Lemma 6.14, one can show that $T_0 F^m A \subset F^{m-2} A$. Hence, on $F^2 A$, $t_j t = t_j(\sum_i t(b_i)t_i)$ for any j . We can calculate $t_j(\sum_i t(b_i)t_i)(cb_j^2)$, where $c \in F^0 A$, in two ways: either by, applying t_i 's first and then multiplying the results by coefficients $t(b_i)$, or directly by applying $\Delta(t)$. It follows from the discussion immediately preceding Lemma 6.19 that $\Delta(t)(cb_j^2) = t(cb_j)b_j + cb_j t(b_j)$. By comparing the results we see that $t(b_j)$ must commute with c . Therefore, $t \in \text{Span}(t_1, \dots, t_m)$ and we are done. \square

Clearly, if a simple pseudoalgebra $\text{Diff } A$ is finite as an H -module, A is finite, hence it must be of type (\star) . Since X^{cop} acts trivially on A , $\text{Diff } A$ is necessarily a current algebra.

Corollary 6.24. *Let R be a simple differential H -pseudoalgebra that is finite as an H -module. Then $R = \text{Cur End}_n(\mathbb{C})$, $n > 0$.*

Remark 6.25. In the case of conformal algebras, this is true without the assumption of differentiability [11]. The same probably holds for the general case since the classification of finite H -pseudoalgebras should follow the classification of finite Lie H -pseudoalgebras (the latter was obtained in [3]).

Now we will describe H -pseudoalgebras $\text{Diff } A$ when A is either of the type $(\star\star)$ or $(\star\star\star)$ as defined in Theorem 6.23. So, let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} , and $H' = U(\mathfrak{h})$. We can consider H' -pseudoalgebra $R' = \text{Diff}_{H'} A$. Recall that in Example 3.12 we introduced a notation for such pseudoalgebras: Cend_n^ϕ (see also Remark 3.13). A simple comparison of (3.1) and (3.5) shows that $\text{Diff}_H A = \text{Cur}_{H'}^H R'$. Therefore, we conclude:

Corollary 6.26. *Let $R = \text{Diff } A$ be a simple differential H -pseudoalgebra such that A is small and $F^0 A$ is simple. Then either $R = \text{Cur End}_n(\mathbb{C})$ or $R = \text{Cur}_{U(\mathfrak{h})}^H \text{Cend}_n^\phi$ for a Lie subalgebra \mathfrak{h} of \mathfrak{g} , $\phi \in H^2(\mathfrak{h})$, and $n > 0$.*

6.5. Proof of Theorem 1.3

Let $\text{Diff } A$ be a simple associative pseudoalgebra. In the previous subsection we classified its underlying algebra A satisfying certain conditions. Our goal here is to “translate” this condition into one for pseudoalgebras. We need to study the properties of current subalgebras of $\text{Diff } A$.

Lemma 6.27. *Let $\text{Diff } A$ be a unital associative pseudoalgebra with the pseudoidentity e . Then among its current unital subalgebras whose pseudoidentity is e , $\text{Cur } F^0 A$ is maximal.*

Proof. Let R be a unital subalgebra of $\text{Diff } A$ with pseudoidentity e . Pick

$$a = \sum_I \partial^I \tilde{a}_I \in R.$$

For J maximal such that $a_J \neq 0$, $a_{I'} e \in R$. By induction, all $\tilde{a}_I \in R$.

Let R be a current unital subalgebra of $\text{Diff } A$ with pseudoidentity e (i.e., $R = \text{Cur } B$ and $e = \tilde{1}$ with regard to the canonical H -basis of $\text{Cur } B$). Let $a = \sum_I \partial^I \tilde{a}_I$ be an element of the canonical basis of $\text{Cur } B$. For a nonzero J , maximal such that $a_J \neq 0$, $a_{I'} e = 0$, hence $a_J = 0$, and $a = a_0$. Since for $I > 0$, $e_{I'} a = 0$, we conclude that $a_0 \in F^0 A$ and $a \in \text{Cur } F^0 A$. \square

Therefore, if $\text{Diff } A$ is simple and its maximal unital current subalgebra with the same pseudoidentity is simple and finite, A is small and X^{cop} -simple, and $F^0 A$ is simple.

By definition, elements \tilde{a} when $a \in F^0 A$ form a (unital) current subalgebra of $\text{Diff } A$. Hence, by Corollaries 6.24 and 6.26, the pseudoalgebras satisfying the conditions of Theorem 1.3 are precisely the ones listed there.

It remains to show that pseudoalgebras from that list satisfy the conditions of the Theorem. Simplicity follows from Lemma 6.1. The maximal unital current subalgebra is finite simple by Lemma 6.27. This completes the proof of Theorem 1.3.

6.6. Small pseudoalgebras

Finiteness of filtration components of A should be translated into a finiteness condition for $\text{Diff } A$. With this in mind, we propose the following

Definition 6.28. A unital differential associative pseudoalgebra is called *small* if all its unital current subalgebras are finite as H -modules.

Conjecture 6.29. A unital pseudoalgebra $\text{Diff } A$ is small if and only if A is a small X^{cop} -simple algebra.

Together with the conjectural statement in Section 6.3, this will imply a stronger version of Theorem 1.3:

Corollary 6.30. A simple unital pseudoalgebra such that all its unital current subalgebras are finite over H is either of the pseudoalgebras from Theorem 1.3.

The proof of this conjecture will require understanding what unital current subalgebras a differential pseudoalgebra may contain and, more generally, the description of the structure of unital subalgebras of a differential pseudoalgebra. The latter is hard to ascertain, as there are some counterintuitive examples: for instance, a current conformal algebra may contain a non-current unital subalgebra [16].

Acknowledgments

I am grateful to J.T. Stafford and E. Zelmanov for fruitful discussions and to V.G. Kac for providing an updated version of [11].

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